Fuzzy Sets & Expert Systems in Computer Eng. (2):

Fuzzy Sets

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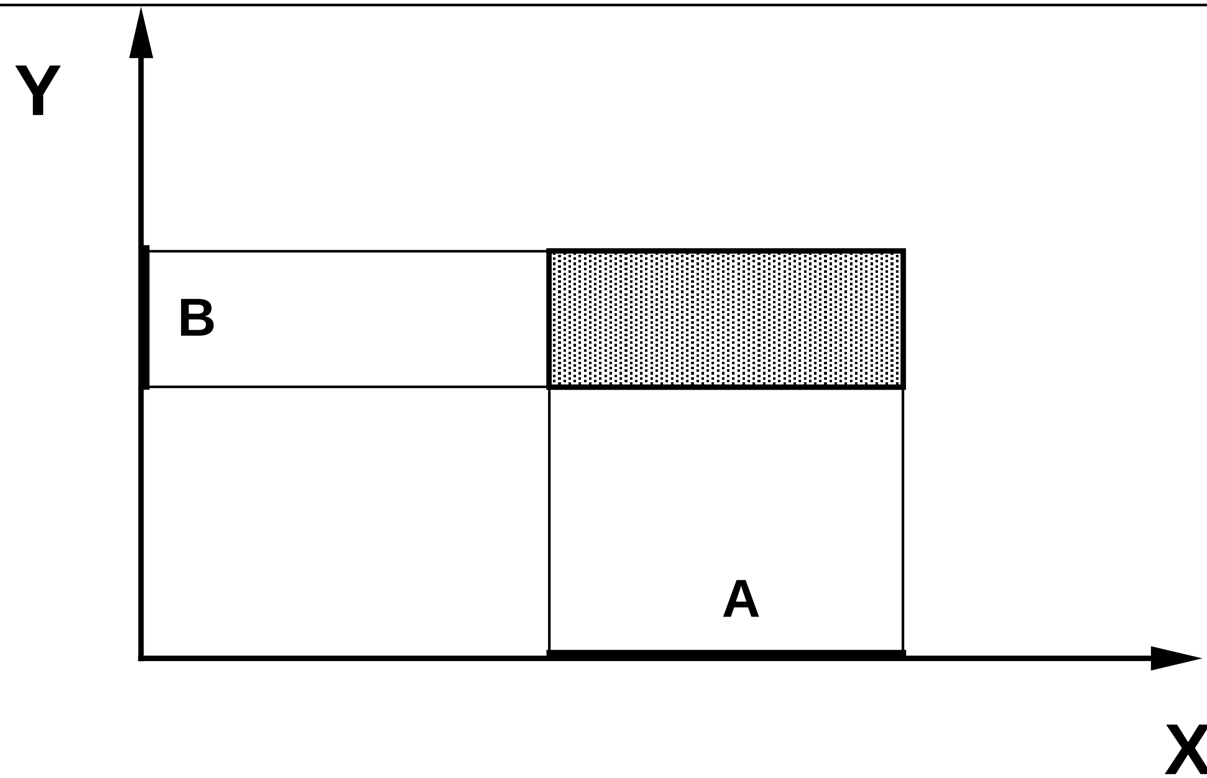
(adapted from slides by Roger Jang and Enrique Ruspini)

Outline

- Conventional Cartesian Product
- Crisp and Fuzzy Relations
  - Characterization
  - Projections & Cylindrical Extensions
- Extension Principle and Fuzzy Numbers
- Compatibility Relations
  - Definition, Composition, Approximation
- Generalized Modus Ponens
  - Mapping: Disjunctive Approximation
  - Implication: Conjunctive Approximation
- Translation rules
- Approximate Reasoning

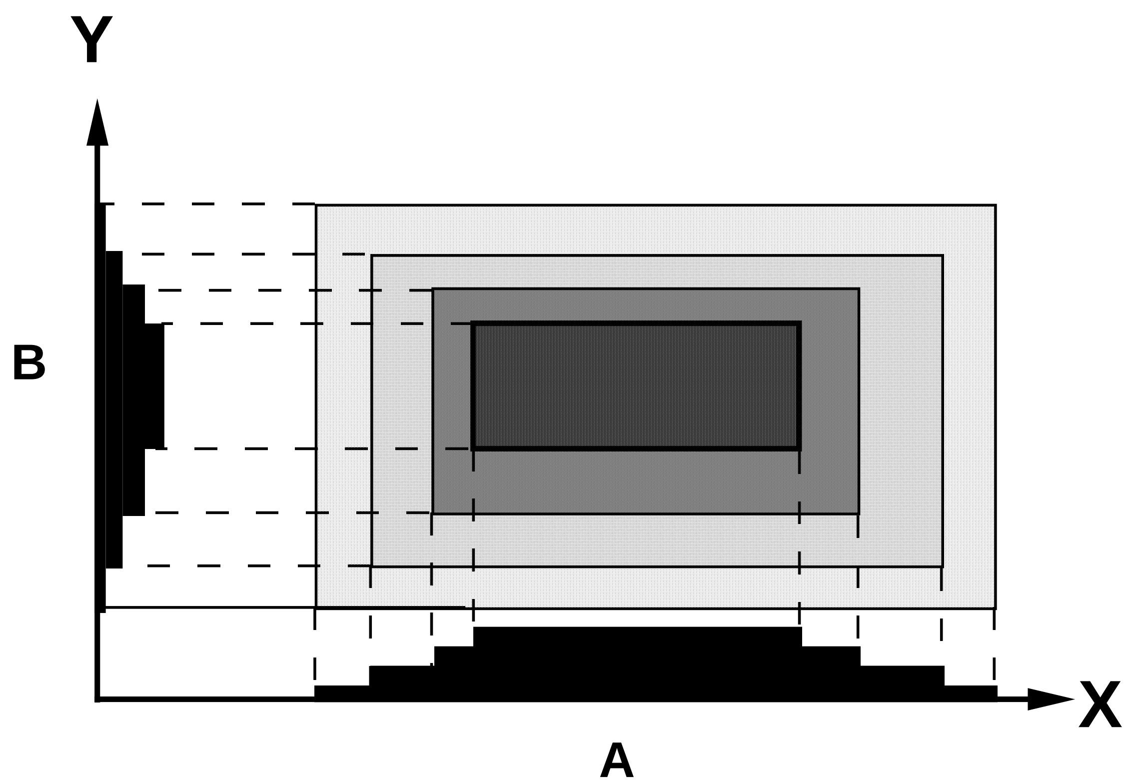
Conventional Cartesian Product

$A \times B$ is the set of all pairs (x, y) with x in A and y in B



The cartesian product is the intersection of cylinder sets

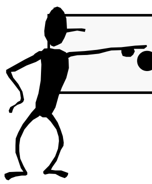
Cartesian Product of Fuzzy Sets



- A: $X \to [0,1]$, B: $Y \to [0,1]$
- $A \times B$: $X \times Y \to [0,1]$: $(x, y) \to \min(A(x), B(y))$

Outline

- Conventional Cartesian Product



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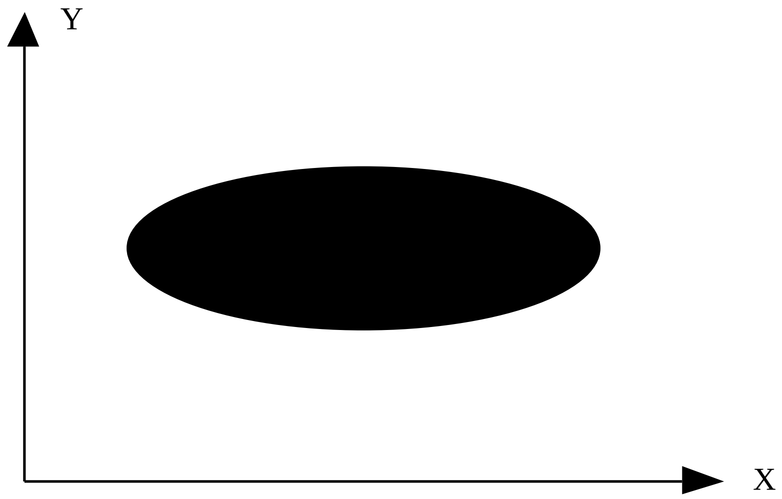
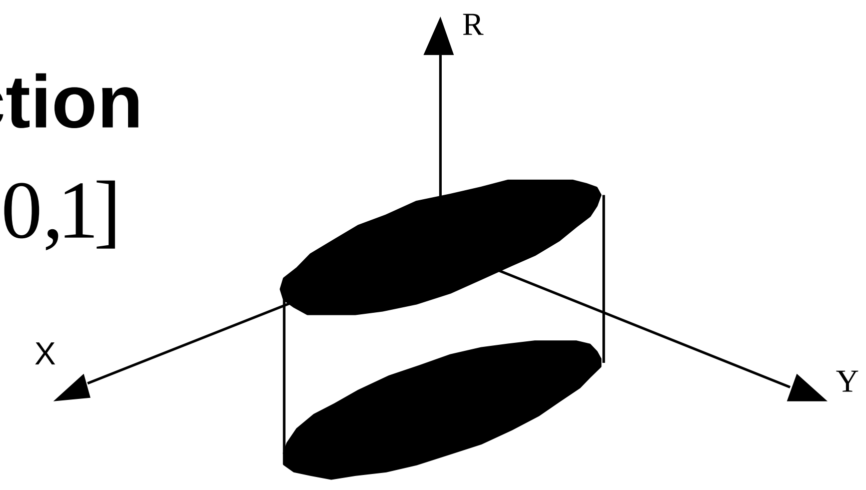
Classical Set Relations



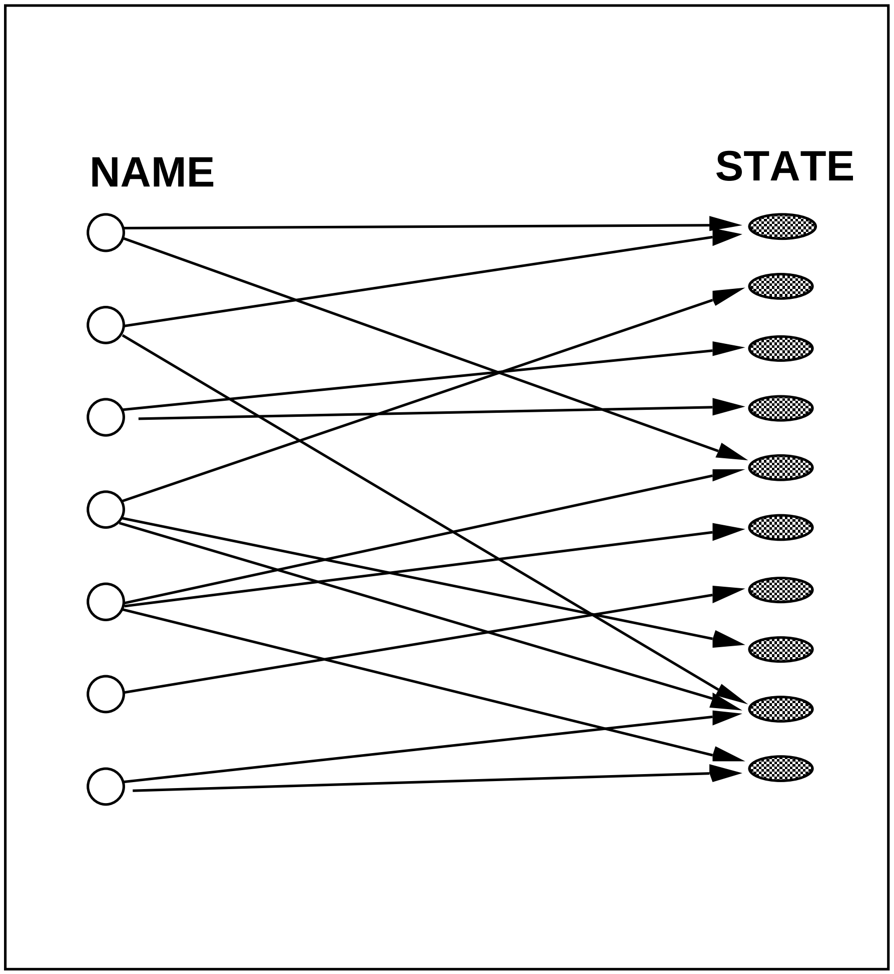
R(X,Y)



Characteristic Function

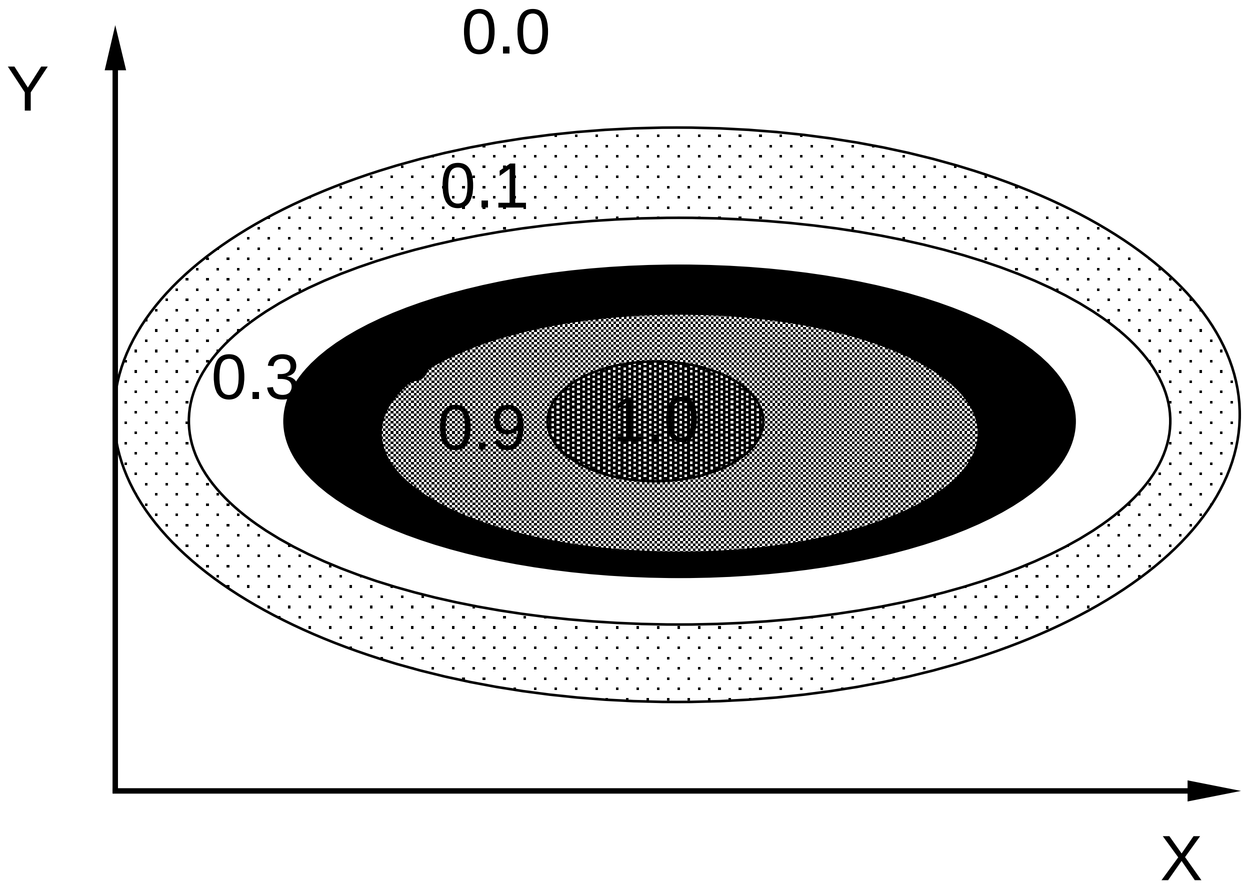
$$R(X,Y):(X,Y) \rightarrow [0,1]$$

Other Representations for Classical Set Relations



NAME	STATE
Washington, G.	VA
Adams, J.	MA
Adams, J.Q.	MA
Jefferson, T.	VA
Madison, J.	VA
... ..	
Lincoln, A.	IL
... ..	

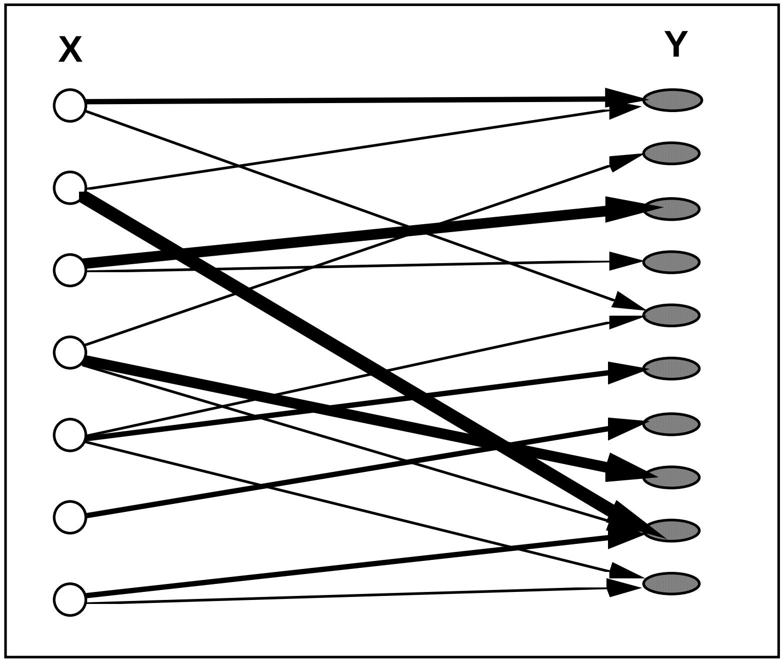
Fuzzy Relations

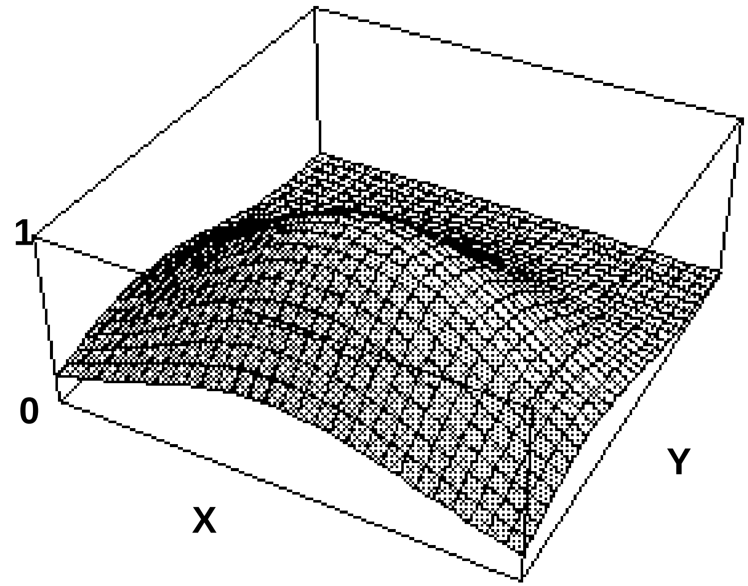


Relation measures the degree by which

x is related to y

Other Representations of Fuzzy Relations



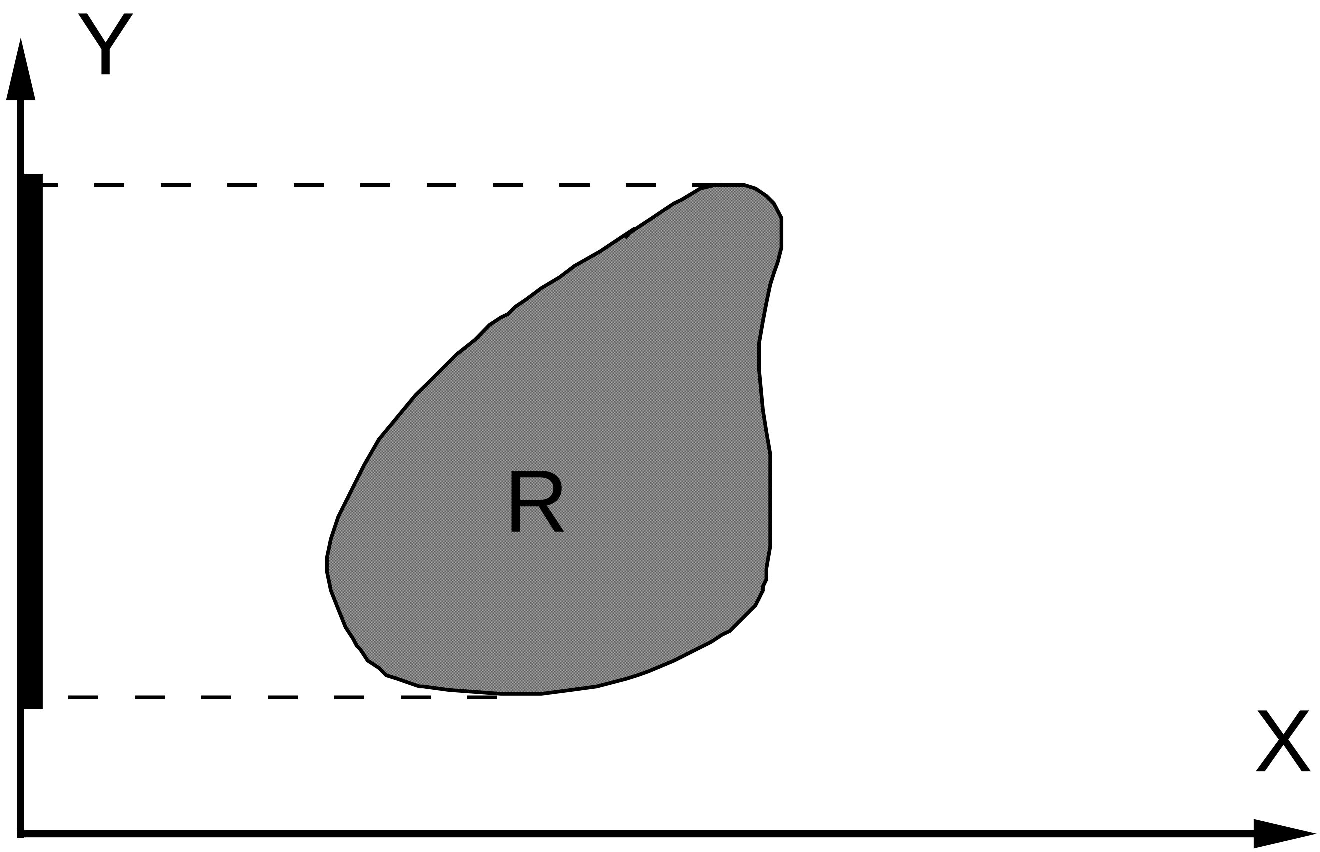


Model 1	Model 2	Similarity
Porsche 921	Porsche 924	1.0
Porsche 921	Corvette	0.8
Toy. Celica	Nissan 380	0.6
RR Silver S.	Ford Escort	0.0
. . .	. . .	. .

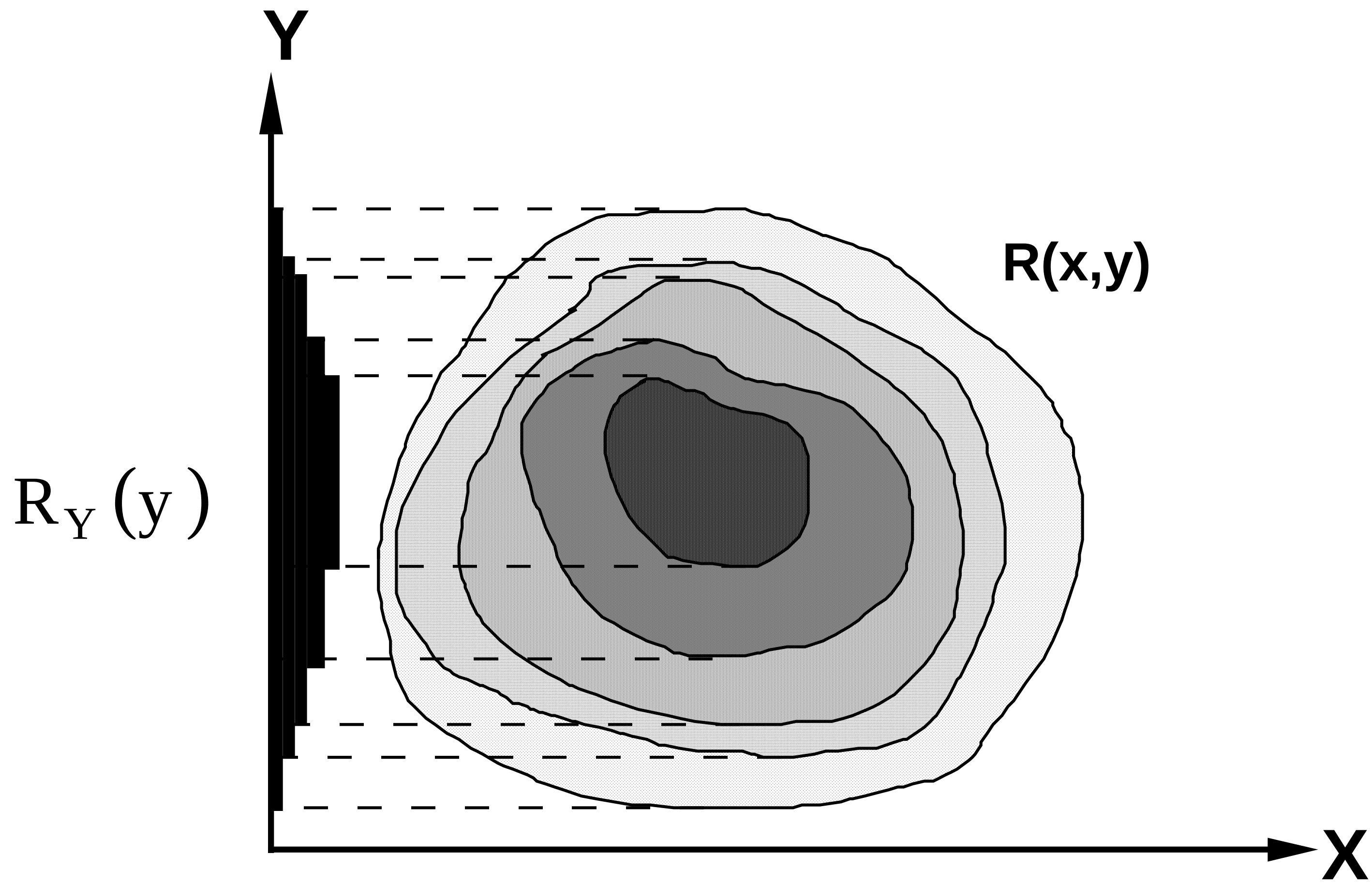
Examples of Fuzzy Relations

- x is *much older* than y *(Ages)*
- x is *larger* than y *(Numbers)*
- x is *much larger* than y *(Numbers)*
- The product of x and y is *approximately* 8 *(Numbers)*
- If the temperature is *high,* then the pressure is *very high* *(Temperature × Pressure)*
- If the angle is *low,* then the acceleration should be *small* *(Angle × Acceleration)*

Conventional Set Projection

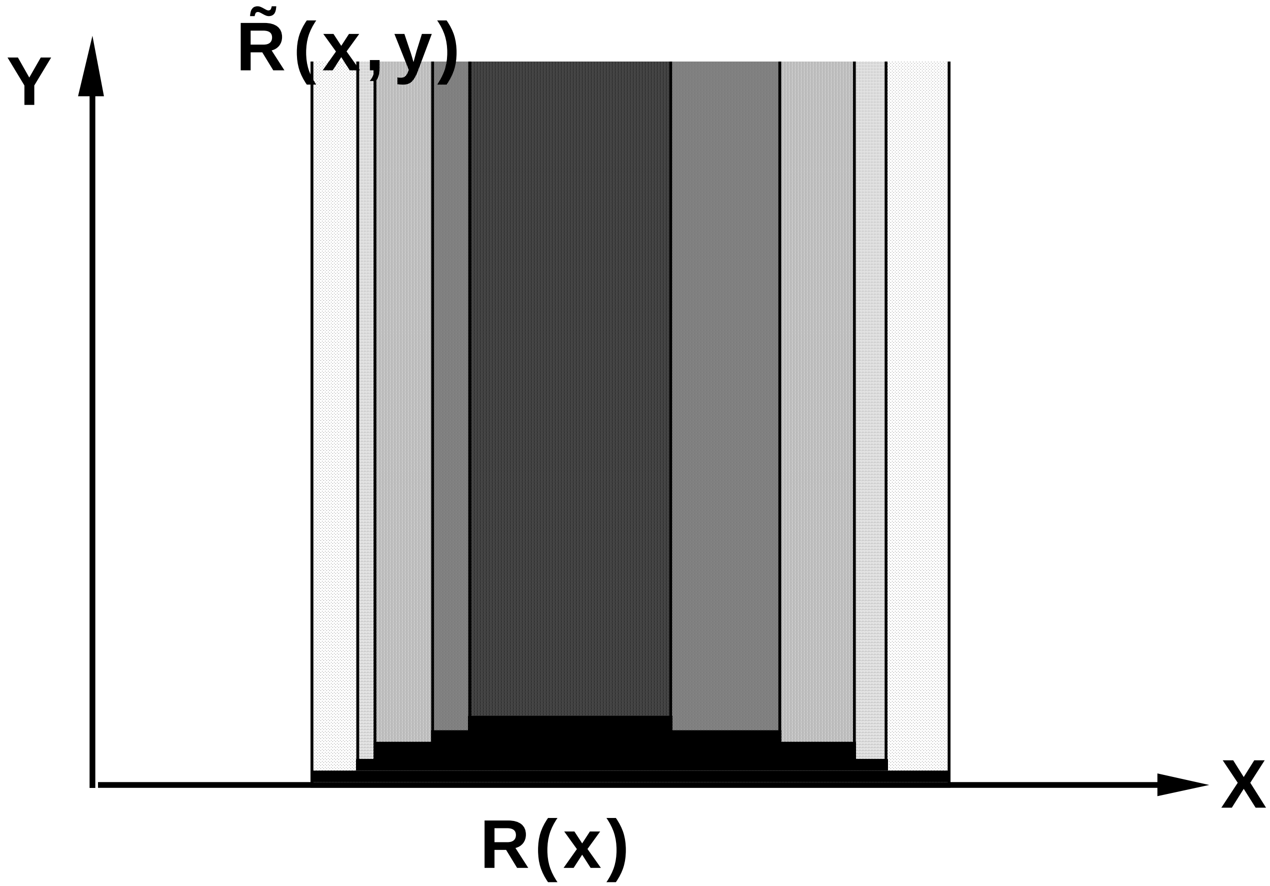


Fuzzy Set Projection



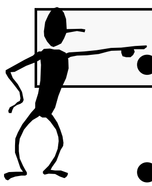
$$R_Y(y) = \max_x [R(x,y)]$$

Cylindrical Extension



$$\tilde{R}(x,y) = R(x), \text{ for all } y$$

Outline



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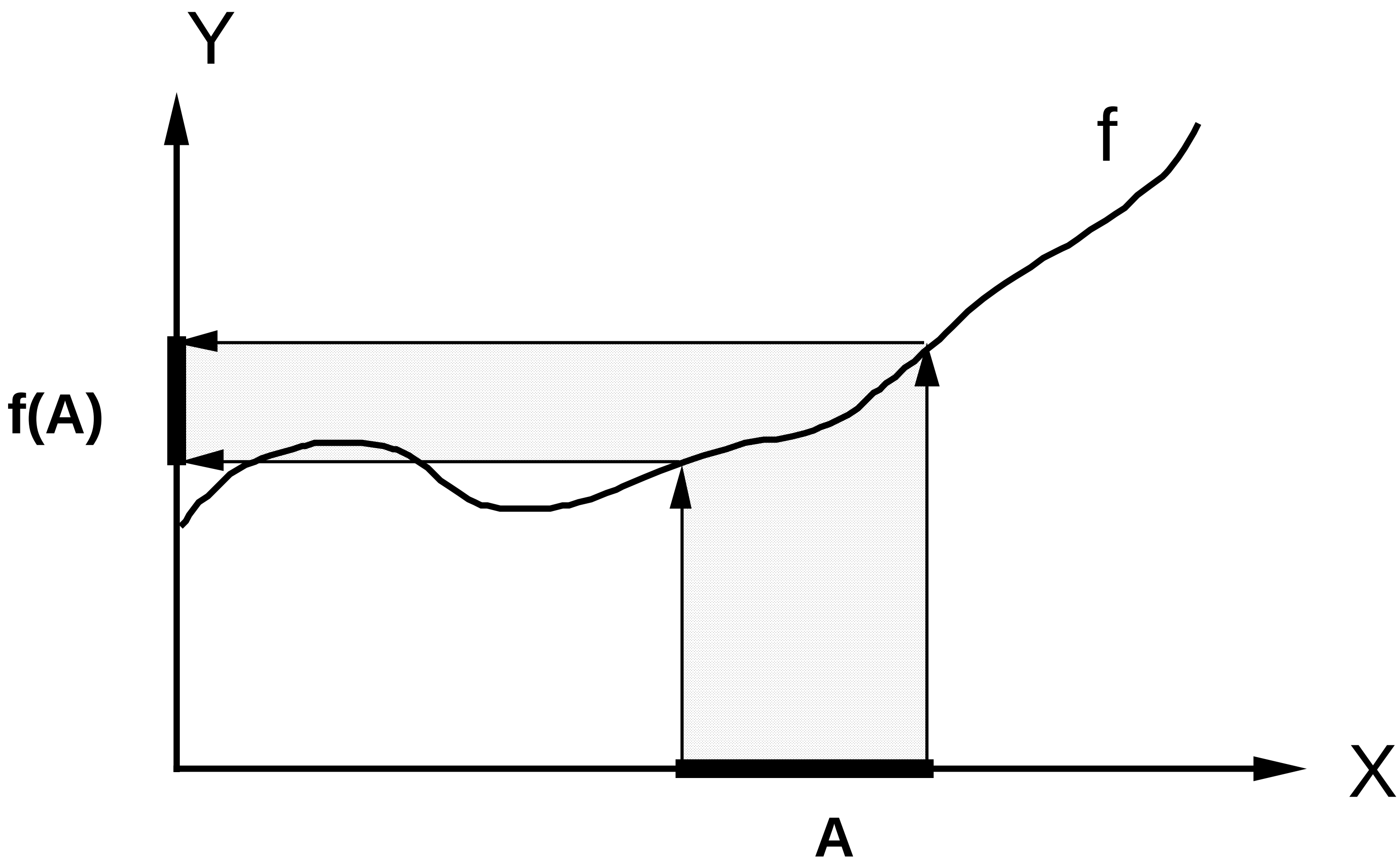
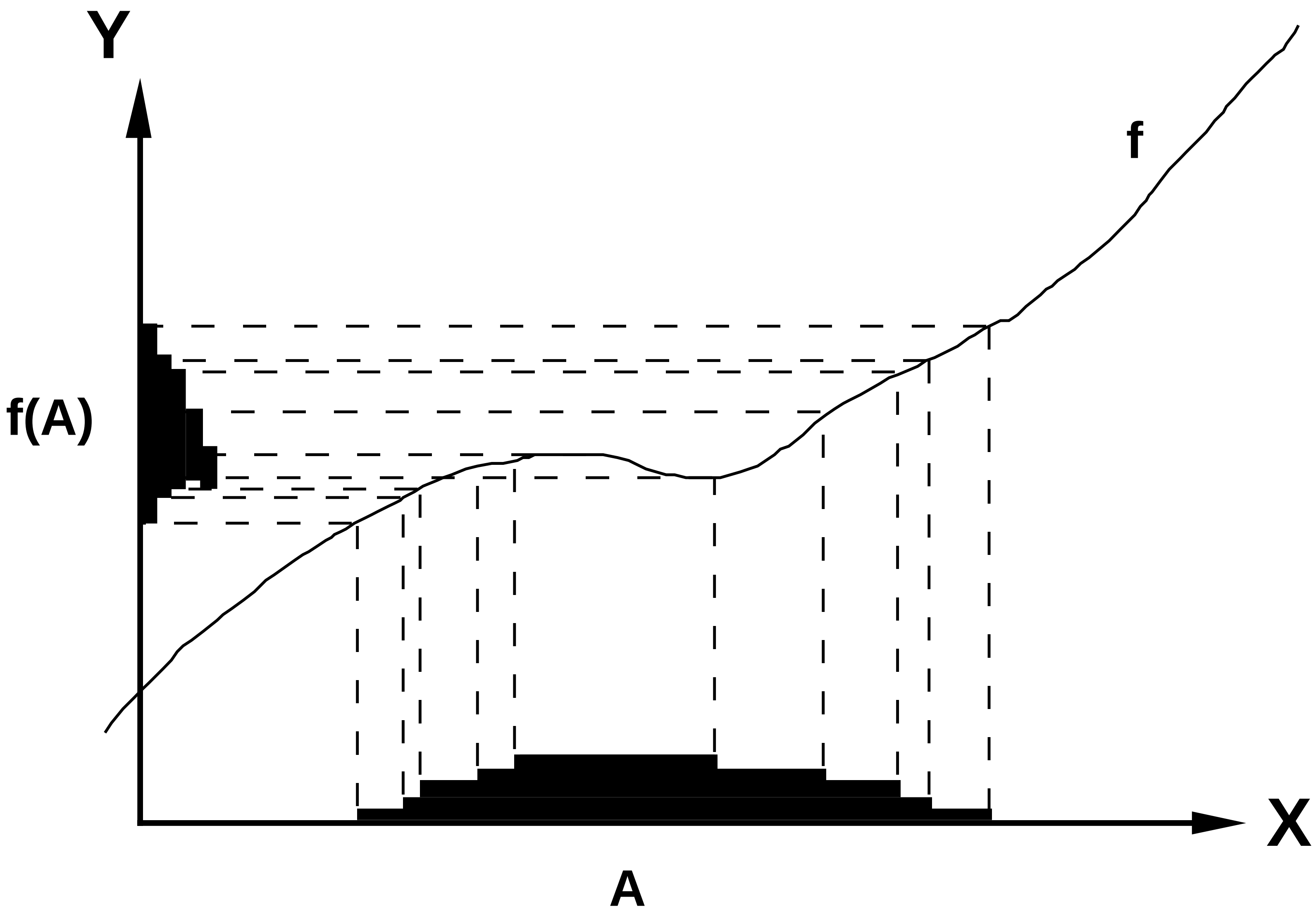
Mapping Conventional Sets



Mapping Fuzzy Sets



The Extension Principle

- Permits generalization of conventional operators
- Based on the generalization of a function

$$\mathbf{f: X \rightarrow Y}$$

into a function mapping fuzzy subsets of **X** into fuzzy subsets of **Y**

- If **x** has a degree of membership μ , then **y = f(x)** is assigned a degree of membership μ

[If more than one **x** is mapped into **y** then the maximum of such memberships is used as the definition of the degree of membership of **y**]

Fuzzy Numbers

- Special fuzzy subsets of the real line
- Examples:
  - Approximately 6
  - Very large
  - Small
- May be combined using generalized operations, e.g.,

$$(A+B)(z) = \sup_{z=x+y} \{ \min [A(x), B(y)] \}$$

The Extension Principle (continued)

$$f: X_1 \times X_2 \times \mathbb{L} \quad X_n \to Y,$$

$$f(A_1, A2, \mathbb{K}, A_n)(y) = \sup_{f(x_1, x_2, \mathbb{L}, x_n)=y} \min_i [A_i(x_i)]$$

<u>Example</u>:

A = 0.1/1 + 0.2/2 + 1/3 + 0.1/4

B = 0.3/1 + 1/2 + 0.5/3

A+B = 0.1/2 + 0.2/3 + 0.3/4 + 1/5 + 0.5/6 + 0.1/7

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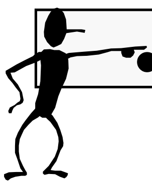

- **Compatibility Relations**
  - **Definition, Composition, Approximation**
- **Generalized Modus Ponens**
  - **Mapping: Disjunctive Approximation**
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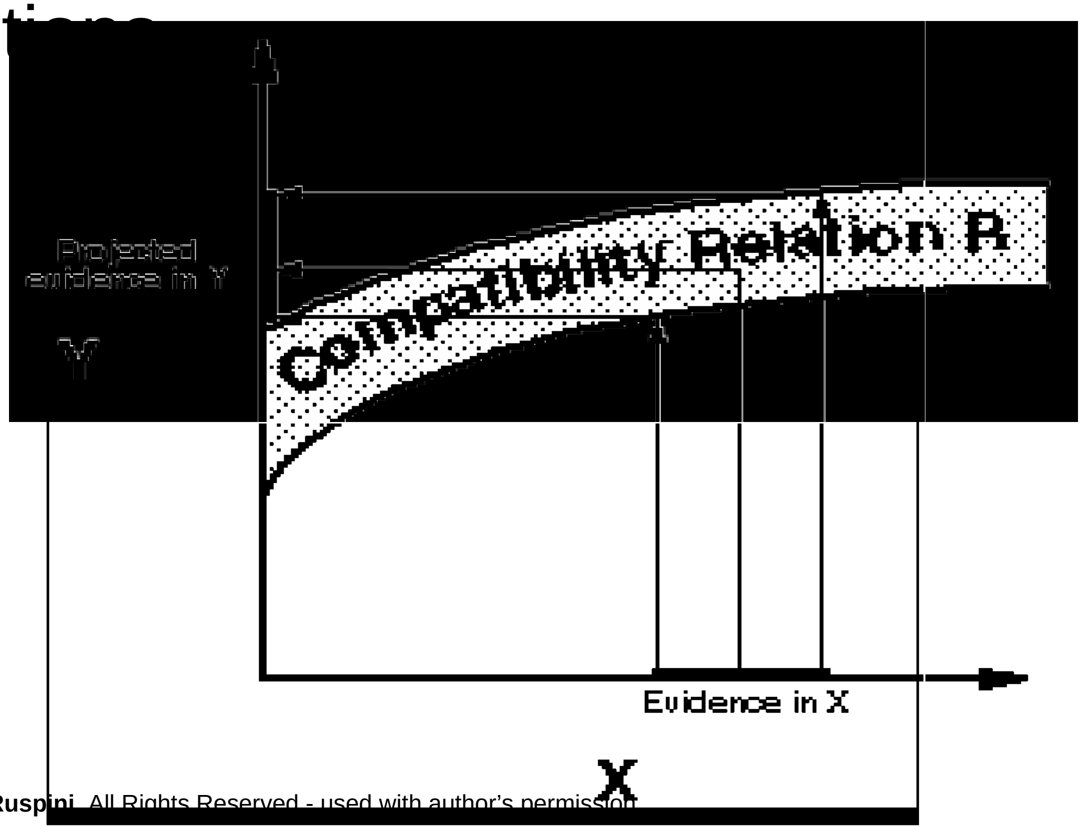
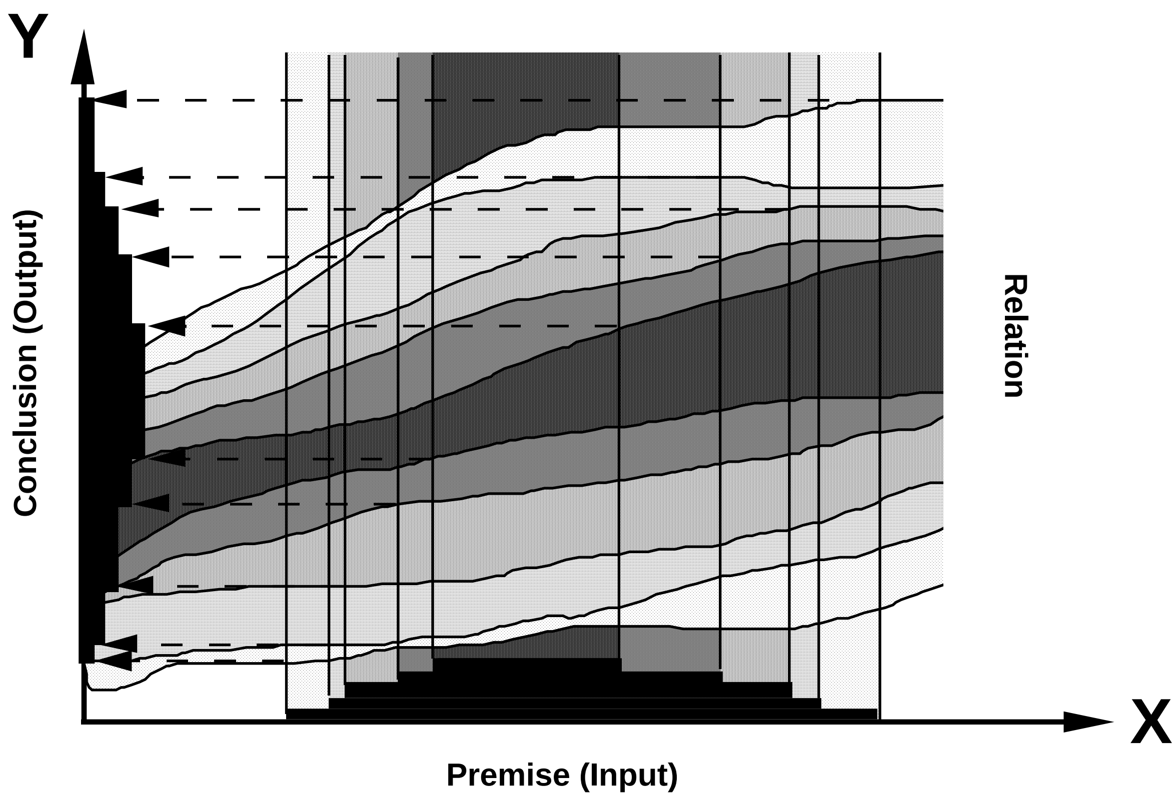
Compatibility Relations

- Define relationship between values of two system variables (in the “actual” world)
- Permits the derivation of possible values of **Y** from knowledge of possible values of **X**
- Defined by means of conditional possibility distributions

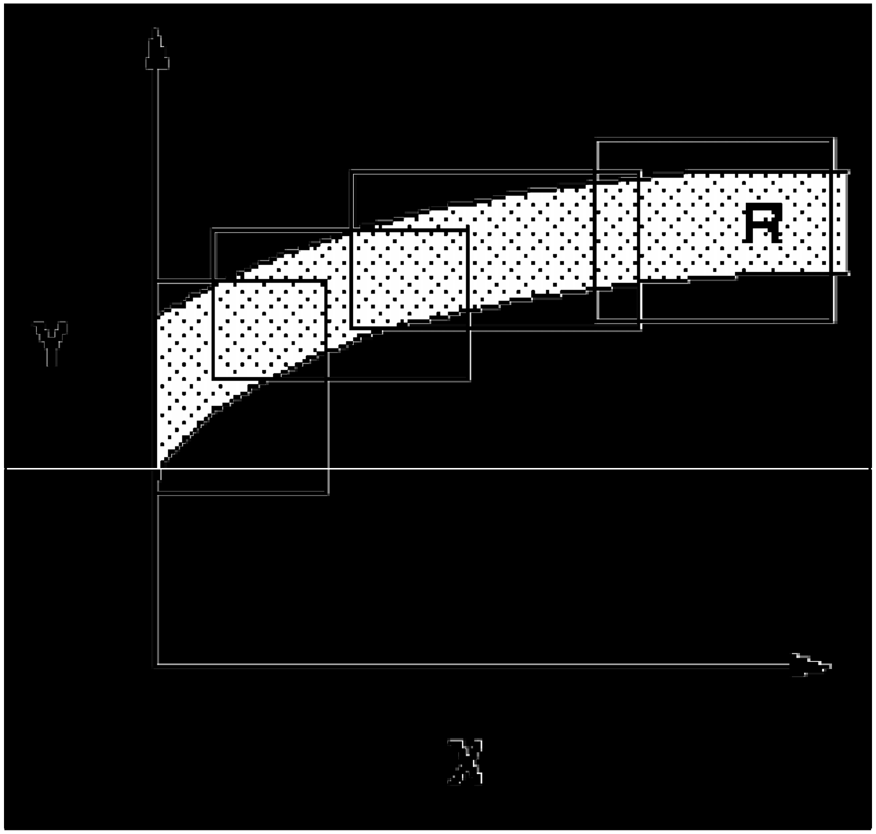


Composition of Sets and Relations

Generalized (Fuzzy) Composition



Approximating Compatibility Relations



- Compatibility Relations may be approximated as a set of inferential rules
- Inferential rules have been given different interpretations

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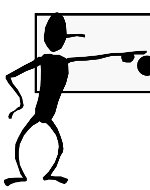

- **Generalized Modus Ponens**
  - **Mapping: Disjunctive Approximation**
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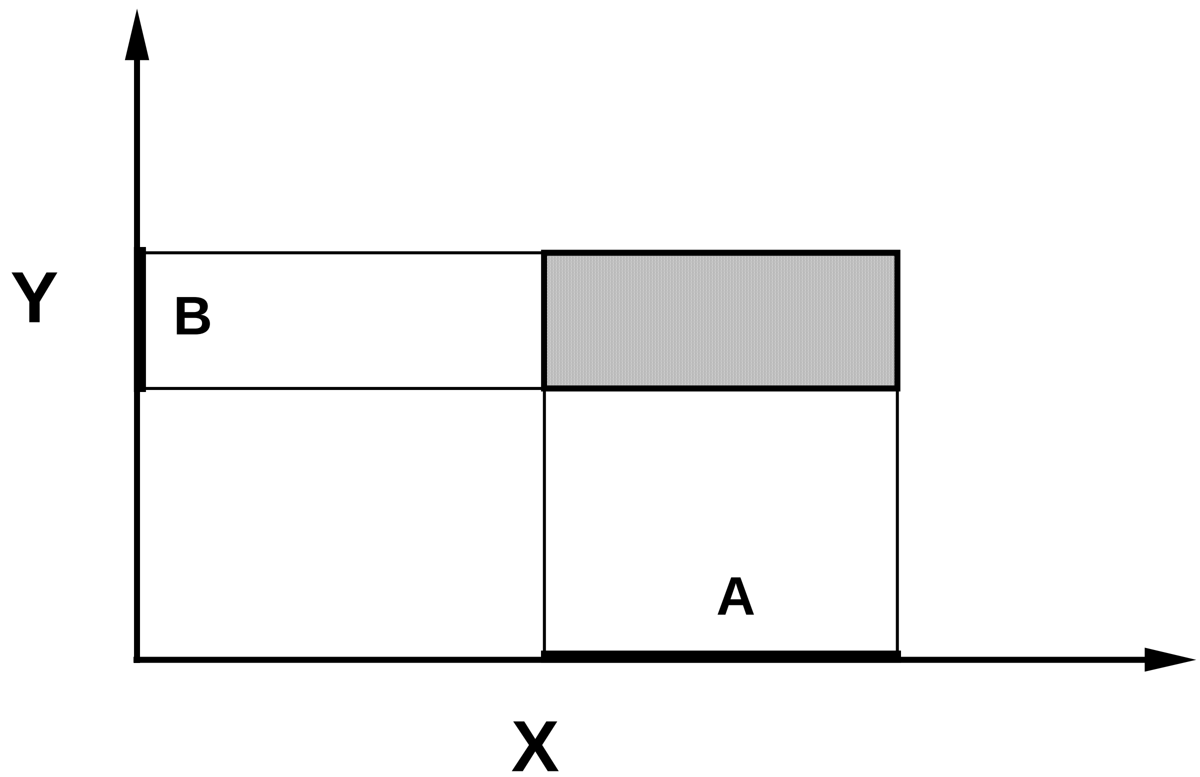
“Mapping” View of Classical Compatibility Relations



Disjunctive Interpretation of a Fuzzy Relation (Zadeh-Mamdani-Assilian)

- “If **X is A**, then **Y is B**” is interpreted as one of a set of regions that must be combined (by union) to approximate the compatibility relation
- Relation is characterized as a “set of points” rather than as the intersection of constraining regions

“If X is A, then Y is B” is modeled by $I(y|x) = A(x) \otimes B(y)$

ZMA Disjunctive Approximants as Fuzzy Relations

Modus Ponens and Generalized Modus Ponens

Classical Modus Ponens:

$$\frac{\begin{array}{l} A \\ A \rightarrow B \end{array}}{B}$$

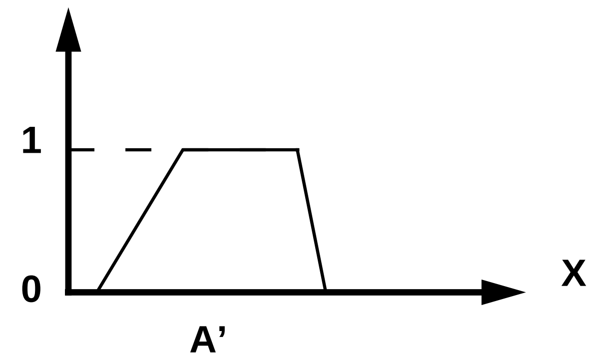
Generalized Modus Ponens:

$$\frac{\begin{array}{l} A' \\ A \rightarrow B \end{array}}{B'}$$

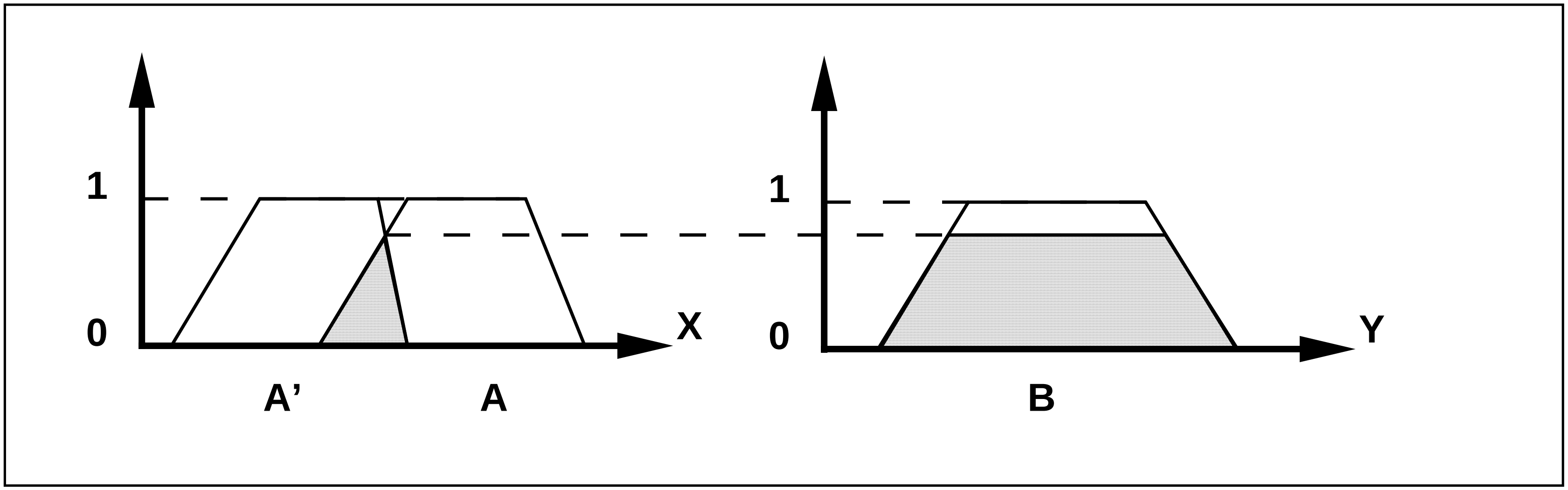
GMP Inputs (ZMA – Single Rule)

- $R \equiv A_X \times B_Y \; , \; (A \rightarrow B)$

- Premise A'_X

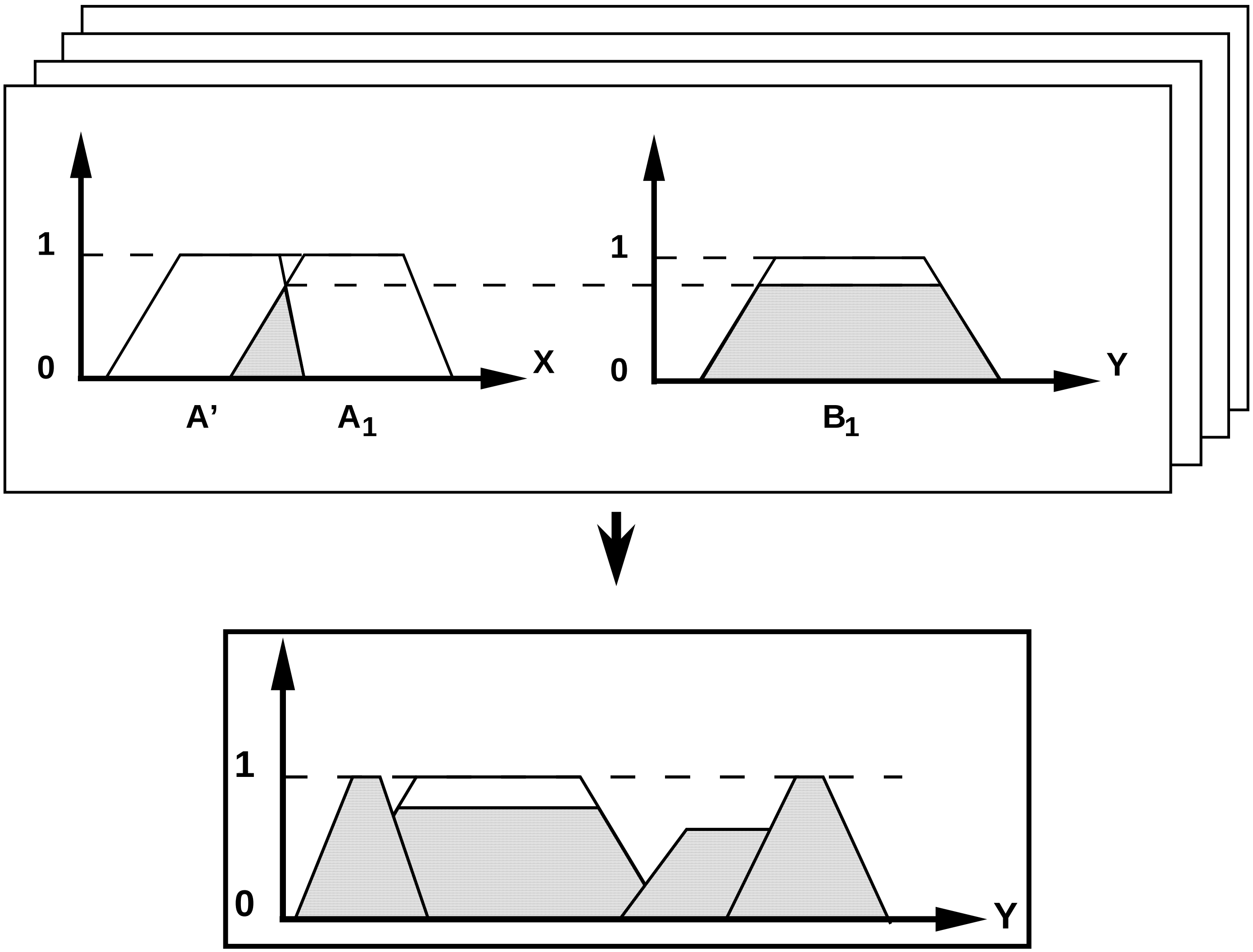


Computation of GMP Output (Single Rule)



The interaction of A and A' determines the influence of B in the conclusion

Computation of GMP Output (Multiple Rules)



Generalized Inference (ZMA)

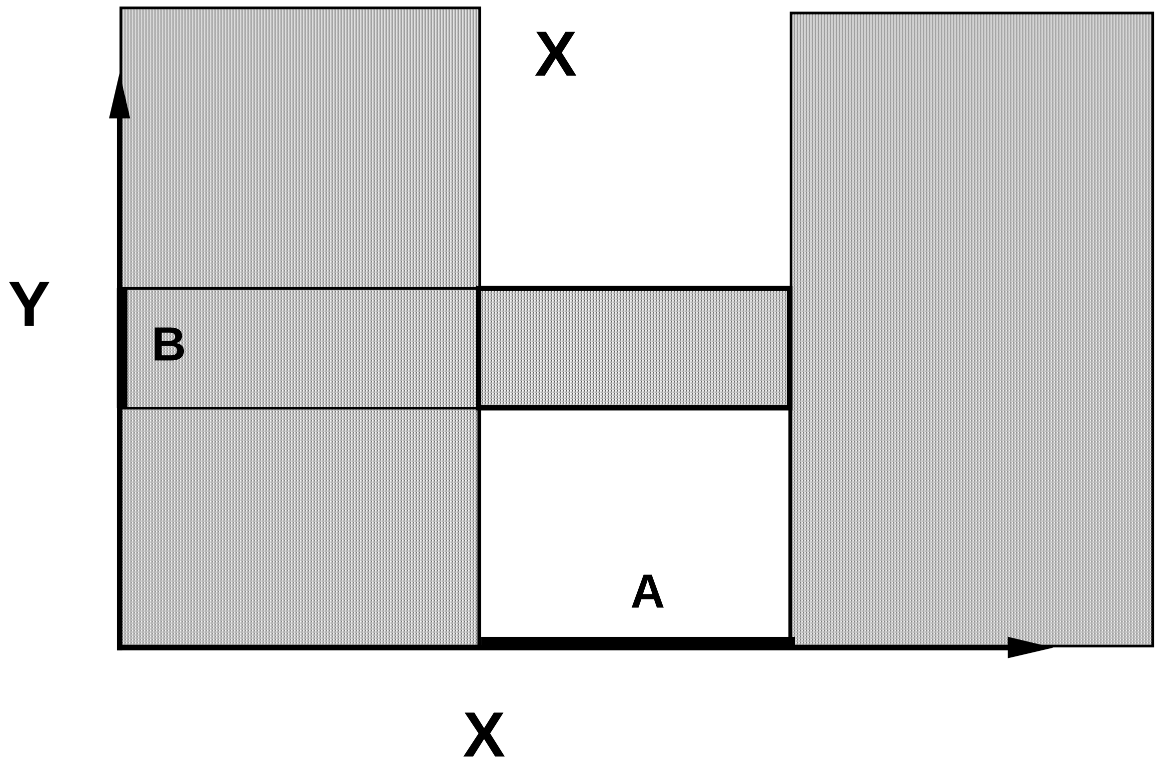
- $R_{XY} = \mathbb{U}_i A_X^i \times B_Y^i$,
- $B_Y' = \mathrm{Proj}_Y \left[R_{XY} \circledcirc A_X' \right] = \mathbb{U}_i \mathrm{Proj}_Y \left[(A_X^i \cap A_X') \times B_Y^i \right]$,
- $B_Y'(y) = \max_i \left[\sup_x \left(\min[\min(A_X^i(x),\ A_X'(x)), B_Y^i(y)] \right) \right]$

The expression

$$\sup_x (\min[\min(A(x),\ A'(x)), B(y)])$$

generalizes the classical inferential rule of *modus ponens*

“Logical” View of (Classical) Conditional Relations



Logical Interpretation of Fuzzy Relations (Zadeh-Trillas-Valverde)

The conditional possibility is an "enclosing" approximation of the compatibility relation

"If X is A, then Y is B" is modeled by $I(y|x) = B(y) \varnothing A(x)$

Inverse of a T-Norm

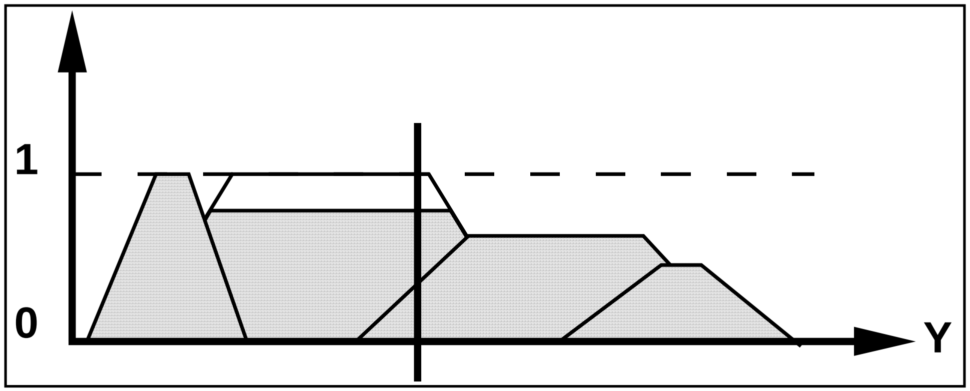
$a \oslash b = \sup \{c : c \otimes b \leq a\}$	
$a \otimes b$	$a \oslash b$
$\max(a + b - 1, 0)$	$\min(1 + a - b, 1)$
ab	a / b, if $b > a$, 1, otherwise
$\min(a, b)$	a, if $b > a$, 1, otherwise

$(q(x) \oslash p(x))$ may be used to measure the degree of inclusion p $(x) \rightarrow q(x)$

ZTV Interpretation as a Fuzzy Relation

Defuzzification

Defuzzification procedures are used to select an adequate decisions among those deemed adequate by the output possibility distribution



Examples of defuzzification:

- Maximum (Mode)
- Centroid (First Moment)

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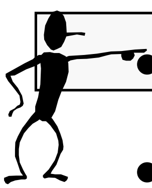


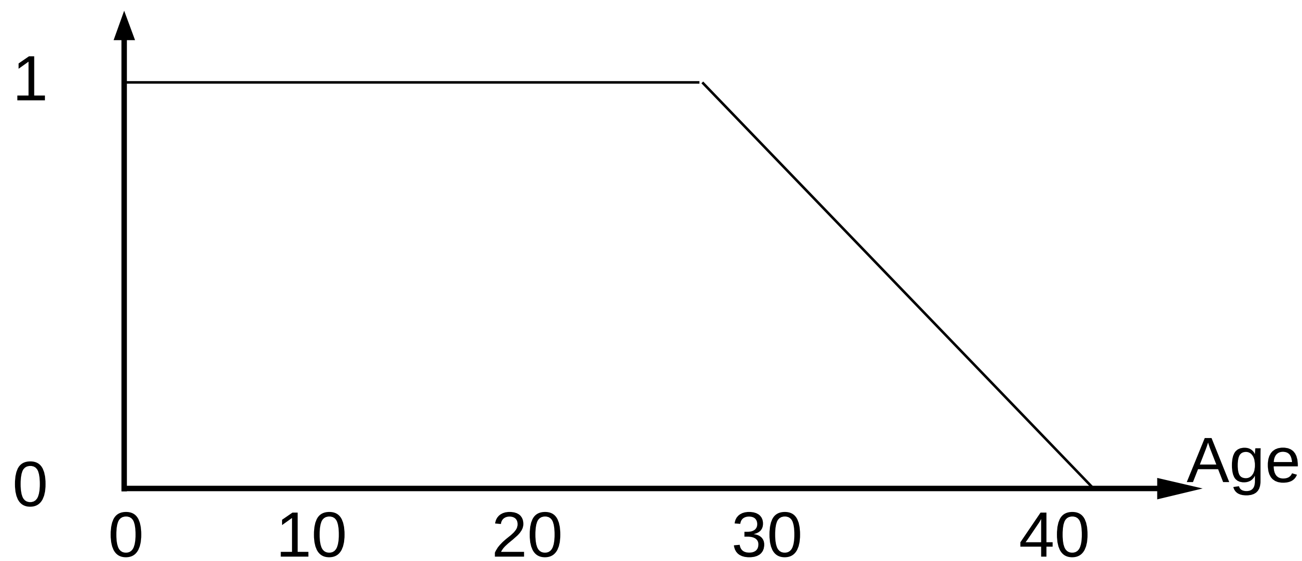
- Translation rules
- Approximate Reasoning

Conventional Formulation of Fuzzy Logic

- Elastic constraints on the values of a variable
- Defined by Possibility Distributions

$$\Pi \textbf{ (Age) = Young}$$

(Assignment of lingüistic variables)



Fuzzy Logic Translation Rules

- $\Pi_{A \cup B}(x) = \max(\Pi_A(x), \Pi_B(x))$,
- $\Pi_{A \cap B}(x) = \min(\Pi_A(x), \Pi_B(x))$,
- $\Pi_{\overline{A}}(x) = 1 - \Pi_A(x)$,
- $\Pi_{\tilde{A}}(x, y) = \Pi_A(x) \quad \textit{for all} \quad y \quad (\textit{Cylindric Extension})$,
- $\Pi_{A \to B}(y|x) = \mu_B(y) \oslash \mu_A(x)$.

Application of Translation Rules

- James is tall <u>and</u> bright
- Vera is either late <u>or</u> lost
- Most students are <u>not</u> rich
- <u>If</u> the weather is rainy, <u>then</u> the probability of catching a cold is high

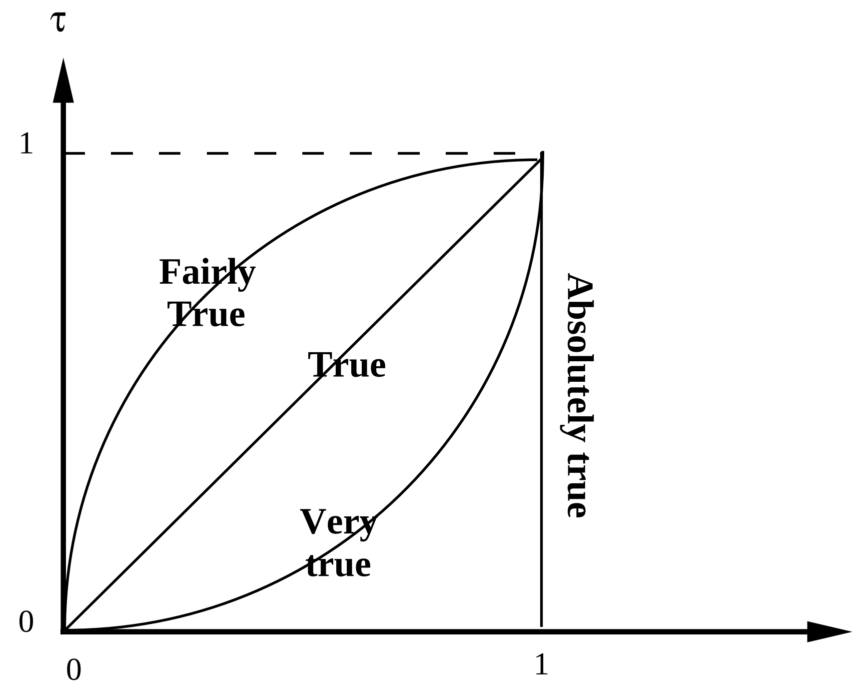
* Don is <u>very</u> worried

** It is <u>very true</u> that Goldilocks eat the porridge

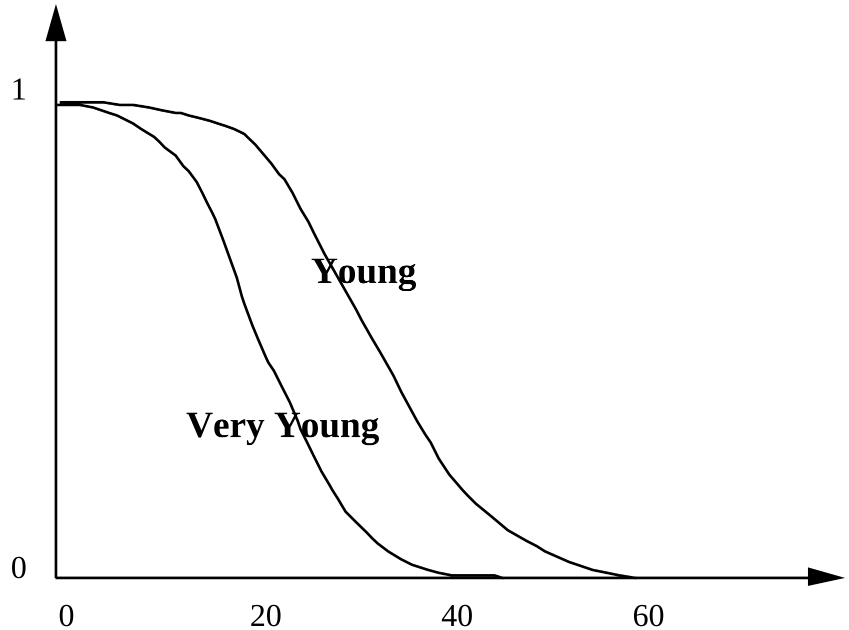
* Linguistic Hedges
** Truth Qualification

Truth Qualification

- **A is B is τ**
- $\tau = A \rightarrow$ **B**
- **Examples:**
  - **John is young is true**
  - **John is young is very true**



Fuzzy Modifiers



- Modifiers are useful to model linguistic hedges

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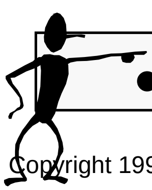


- **Approximate Reasoning**

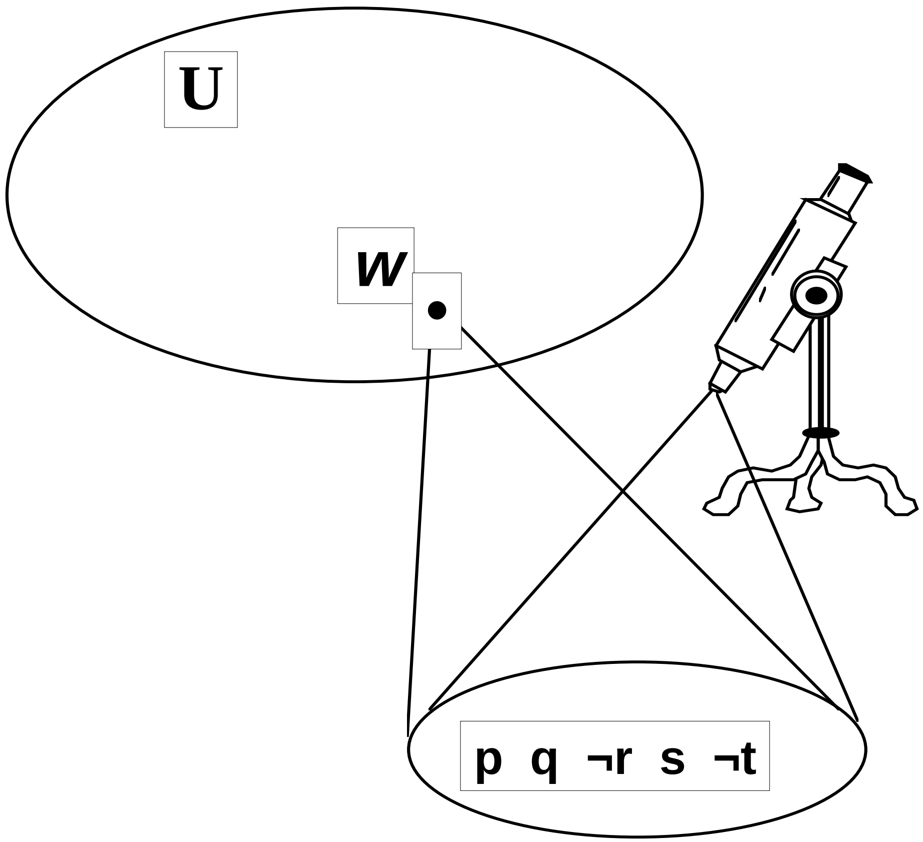
Possible Worlds

- Possible States, Behaviors, Trajectories of a Conceptual System

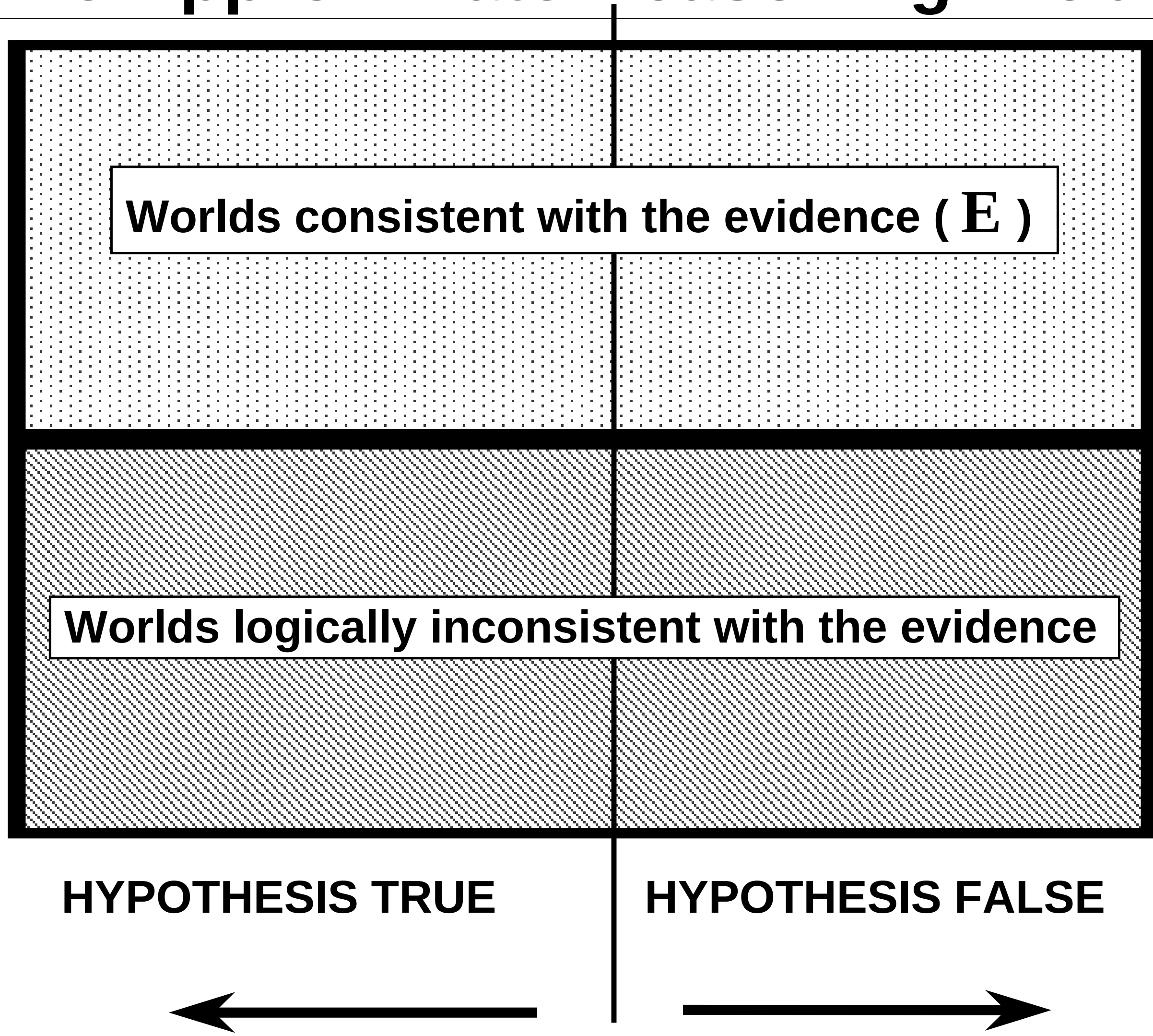
  Examples: Weather System, Vehicle Control System, Portfolio Status

  - Formally equivalent to a Valuation:
  - Assignment of truth-values (i.e., **T**, **F**) to all relevant propositions about the state of system
  - Consistent with rules of logic
- Universe ≡ Set of all Possible Worlds

Possible Worlds as Sets of Propositions

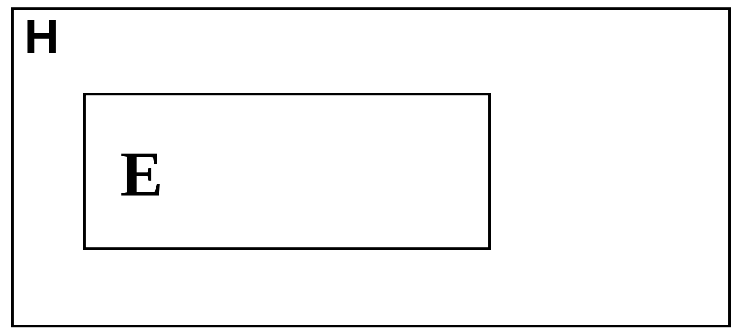


The Approximate Reasoning Problem

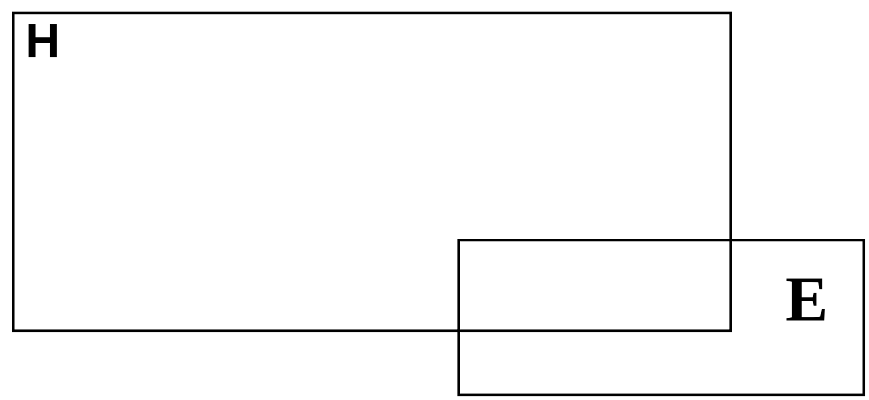


Quantifying Inclusion

Classical Logic:

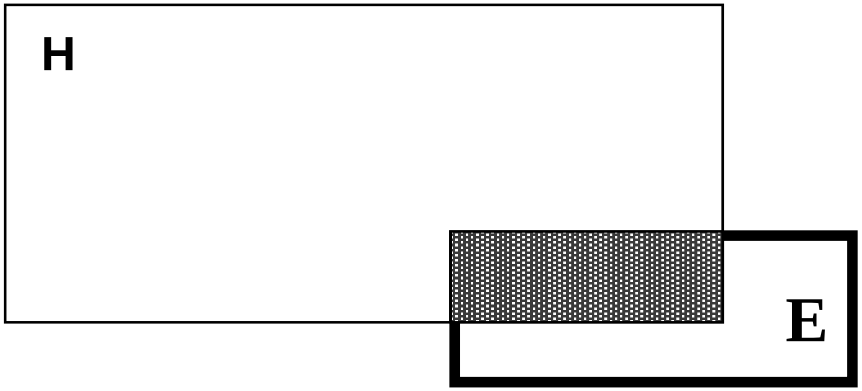


Approximate Reasoning:



Probability and Possibility

Probabilistic Reasoning
(Set Measures):



Possibilistic Reasoning
(Similarities, Distances):