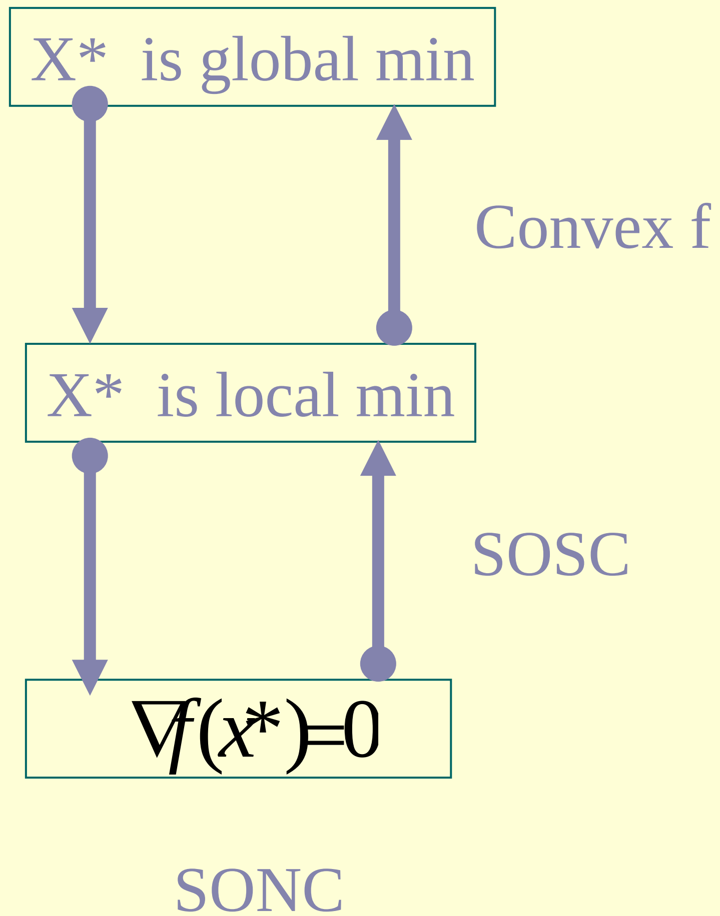
Computational Optimization

Constrained Optimization

Part 2

Optimality Conditions Unconstrained Case



Easiest Problem

- Linear equality constraints

$$\min f(x) \qquad f \in R^n$$

$$s.t. \quad Ax = b \qquad A \in R^{m\times n},\ b \in R^m$$

Assume world is smooth

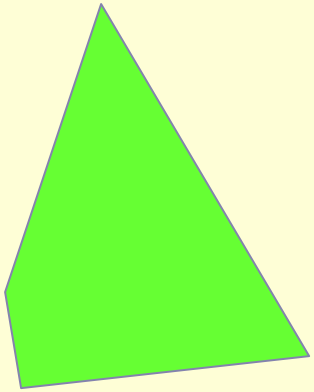
Objective and constraints are twice continuously differentiable for the purpose of this lecture.

Next Easiest Problem

- Linear equality constraints

$$\min\ f(x) \qquad f \in R^n$$

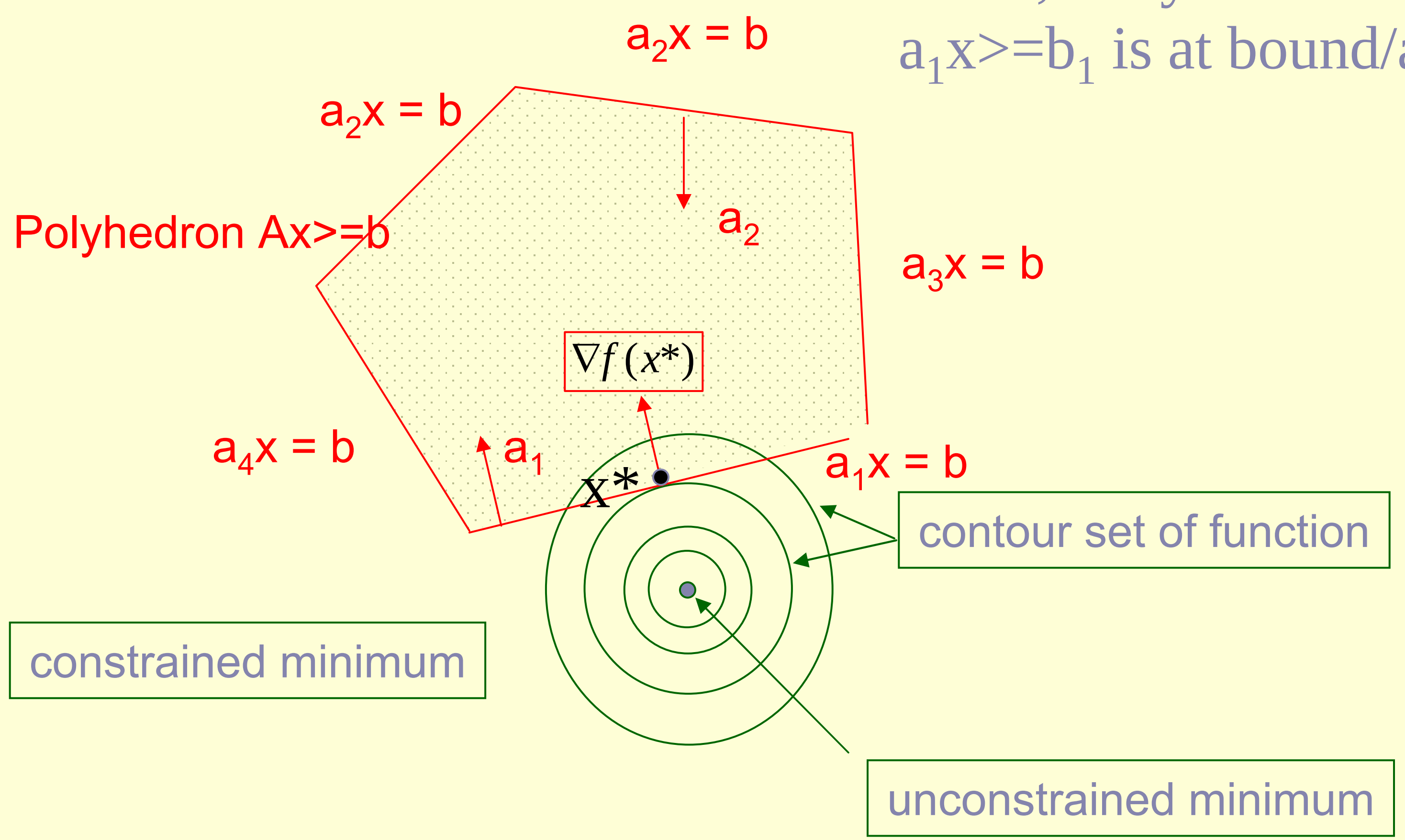
$$s.t.\ \ Ax \geq b \qquad A \in R^{m \times n},\ b \in R^m$$



Constraints form a polyhedron

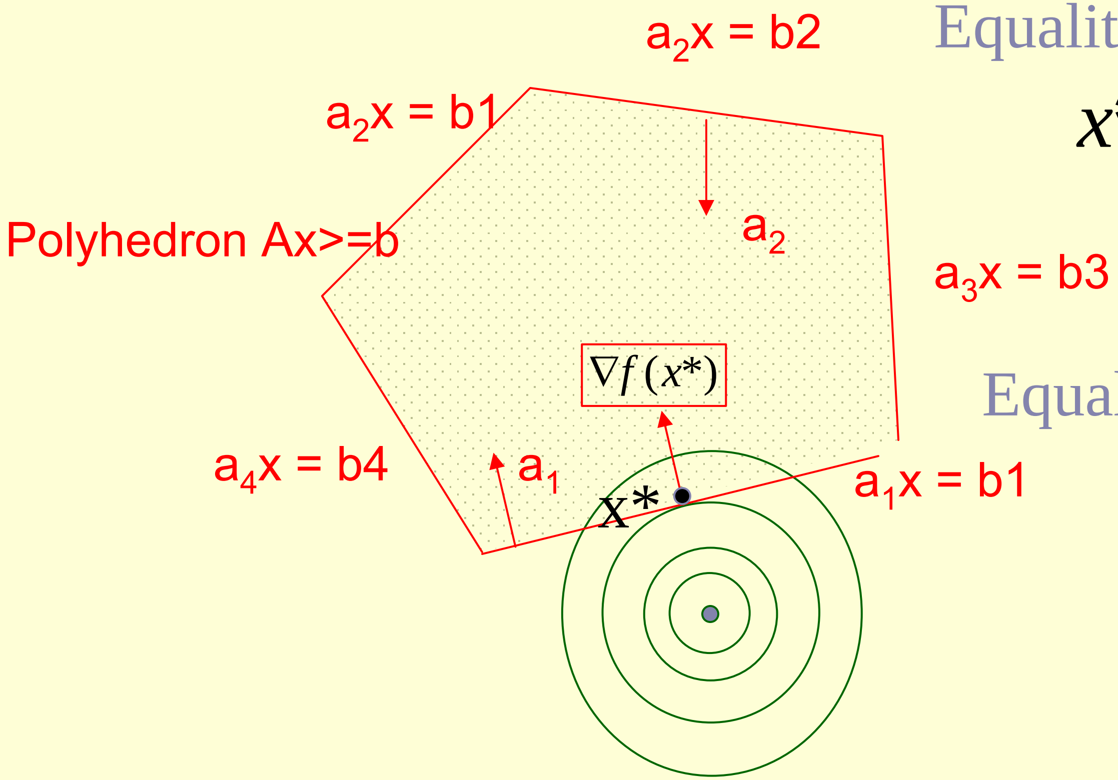
Only some constraints are active.

At x*, only the constraint $a_1x>=b_1$ is at bound/active.



At optimal solution, x*, only one constraint is active.

In this example, the problem is equivalent to minimizing over active set (if you knew it)



Equality FONC:

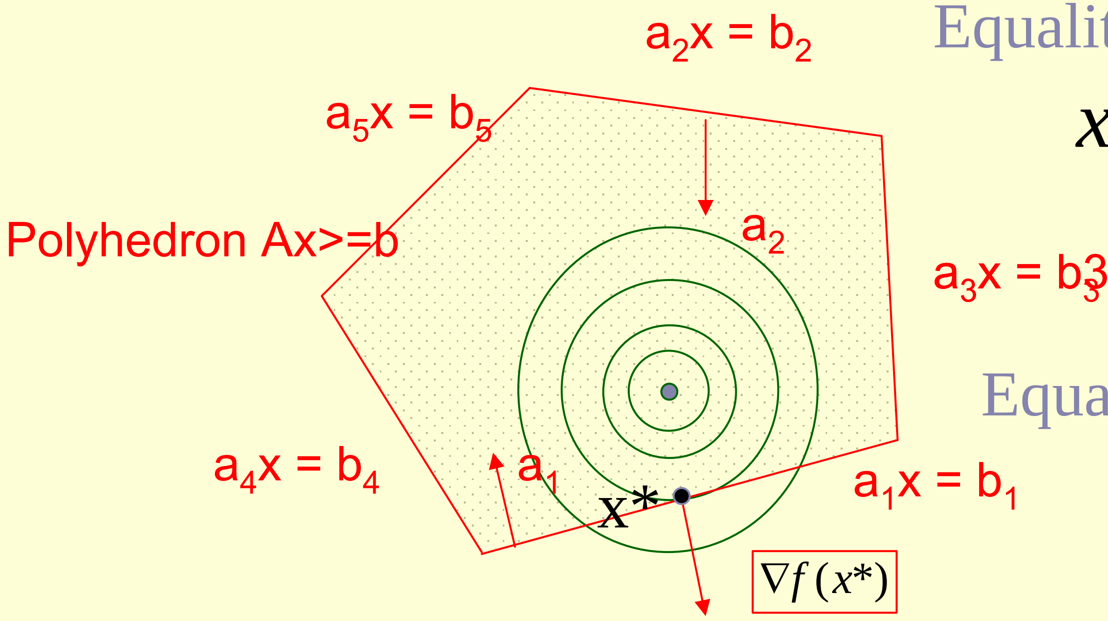
$$x^* \in \min f(x)$$
$$s.t. \;\; a_1 x = b_1$$

Equality FONC:

$$\nabla f(x^*) = \lambda_1 a_1$$
$$a_1' x^* = b_1$$

Note $\lambda_i >= 0$

But in this case x* doesn't solve the inequality problem



Equality FONC:

$$x^* \in \min f(x)$$

$$s.t. \;\; a_1 x = b_1$$

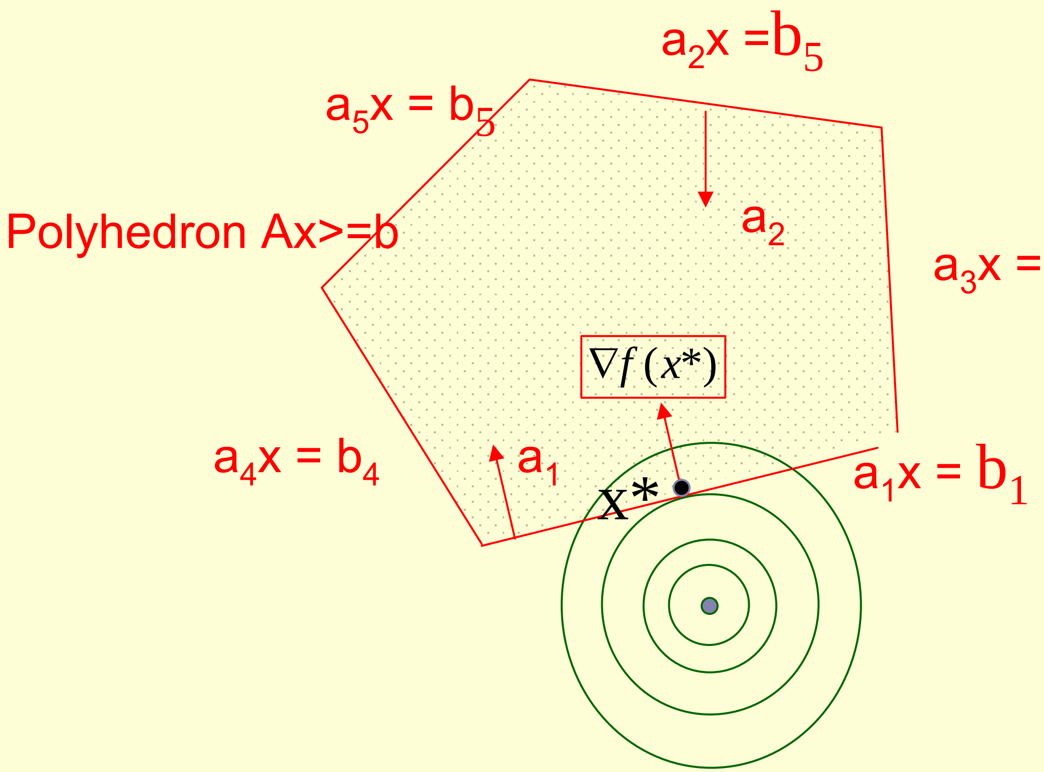
Equality FONC:

$$\nabla f(x^*) = \lambda_1 a_1$$

$$a_1' x^* = b_1$$

Now λ_1 <0. SIGN OF λ_1 matters!

Inequality Case



Inequality problem

$$x^* \in \min f(x)$$

$$s.t. \;\; a_i x \geq b_i$$

$$i = 1 \ldots d$$

Inequality FONC:

$$\nabla f(x^*) = A^T \lambda^* = \sum_i \lambda_i a_i$$

$$Ax^* \geq b$$

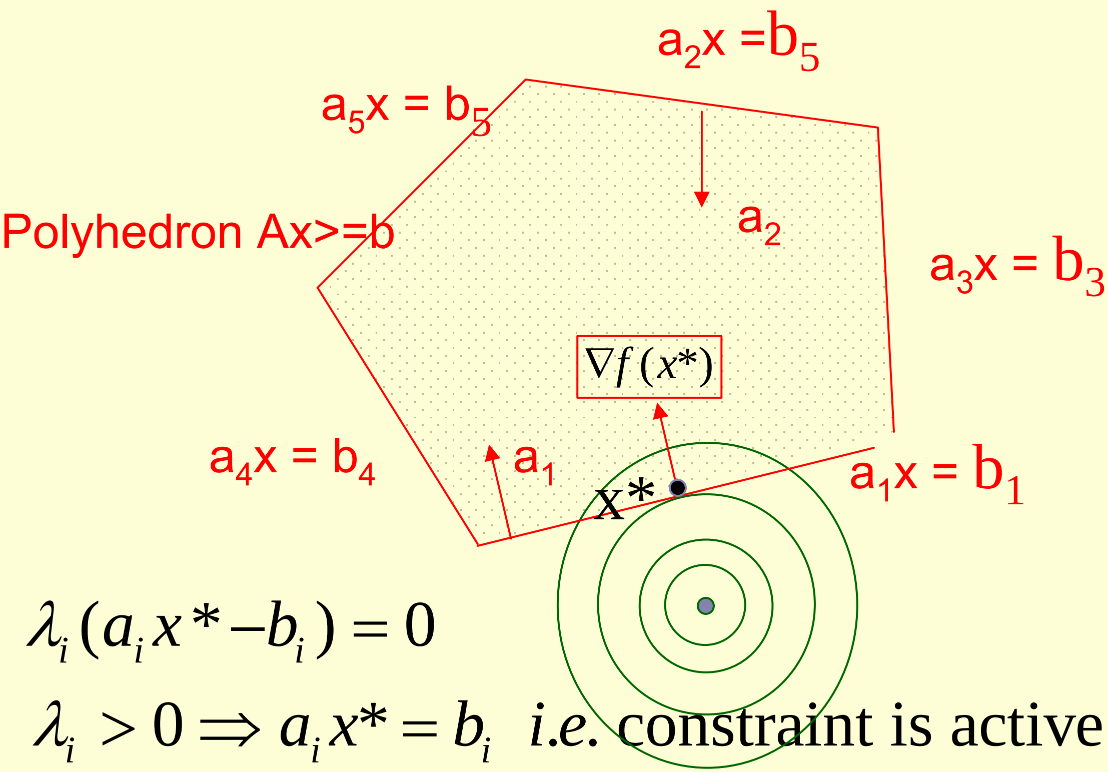
$$\lambda_i (A_i x^* - b_i) = 0$$

$$\lambda \geq 0$$

Nonnegative Multipliers imply gradient points to the greater than Side of the constraint.

Complementarity Condition

Inequality FONC:



$$\lambda_i (a_i x^* - b_i) = 0$$

$\lambda_i > 0 \Rightarrow a_i x^* = b_i$ *i.e.* constraint is active

$\lambda_i = 0 \Rightarrow$ constraint doesn't matter. It can be dropped without changing result.

Lagrangian Multipliers

$Ax \geq b$

$if \quad \lambda_i > 0$

$then \; a_i x = b_i$

λ_i represents sensitivity

If $\lambda_{\mathrm{i}} > 0$ then increasing $\mathrm{a}_i \hat{x} > b_i$ causes the objective to increase.

If b_{i} is changed to make feasible region bigger then

the objective will decrease.

First Order Necessary Conditions for Linear inequality Constraints

- If x* is a local min of f over {x|Ax≥b}, then for some vector λ*

$$\left.\begin{aligned} &\Rightarrow \nabla f(x^*) - A'\lambda^* = 0 \\ &\quad Ax^* \geq b \\ &\quad \lambda^* \geq 0 \\ &\lambda^{*\prime}(Ax^* - b) = 0 \end{aligned}\right\rangle KKT$$

.

Second Order Necessary Conditions for Linear Constraints

- If x* is a local min of f over {x|Ax≥b}, and Z is a null-space matrix for active constraints then for some vector λ*

$$\left.\begin{aligned} &\Rightarrow \nabla f(x^*) - A'\lambda^* = 0 \\ &\qquad Ax^* \geq b \\ &\qquad \lambda^* \geq 0 \\ &\lambda^{*\prime}(Ax^* - b) = 0 \end{aligned}\right\rangle KKT$$

$$\text{and } Z'\nabla^2 f(x^*)Z \text{ is } p.s.d.$$

Z for FONC

- Z is null space matrix for matrix of active constraints at x*
- Index set of Active Constraints

$$I = \{ i \mid a_i x^* = b_i \}$$

- Matrix of active constraints

$$A_I \text{ is } A \text{ restricted to rows of } A \text{ in } I$$

$$Z = null(A_I)$$

Intuition

Every d satisfying

$$\nabla f(x)'d < 0 \qquad Ad \geq 0$$

is a feasible descent direction at x.

The FONC make sure that this set is empty.

$$\nabla f(x)'d < 0 \qquad Ad \geq 0 \text{ has no solution d}$$

$$\text{if and only if}$$

$$\nabla f(x) = A'\lambda \quad \lambda \geq 0 \; has\ a\ solution$$

Due to Farkas's Lemma

Second Order Sufficient Conditions for Linear Inequalities

- If (x*,λ*) satisfies

$Ax^* \geq b$ Primal feasibility

$\left.\begin{array}{l}\nabla f(x^*) = A'\lambda^* \\ \lambda^* \geq 0\end{array}\right\rangle$ Dual feasibility

$\lambda^*{}'(Ax^* - b) = 0$ Complementarity

and SOSC $Z_+{}'\nabla^2 f(x^*)Z_+ \quad is\ p.d.$

Then x* is a strict local minimizer

Sufficient Conditions for Linear Inequalities

where Z_+ is a basis matrix for Null(A_+) and A_+ corresponds to nondegenerate active constraints)

i.e.

$$A_+ = A_J$$

$$\{ j \mid A_j x^* = b_j, \lambda_j^* > 0 \}$$

Sufficient Example

- Find solution and verify SOSC

$$\begin{aligned} \min \quad & -1/2(x_1+1)^2 + 1/2\,x_2^2 \\ s.t. \quad & 0 \le x_1 \le 1 \end{aligned}$$

$$x^* = [1,0]' \qquad \lambda^* = [2,0]$$

Linear Inequality Constraints - I

$$\min \quad -\frac{1}{2}\left(x_1+1\right)^2+\frac{1}{2}x_2^2$$
$$\text{s.t.} \quad 0 \le x_1 \le 1$$

$$\nabla f(x^*) = \begin{bmatrix} -\left(x_1^*+1\right) \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\nabla^2 f(x^*) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Put in standard form :

$$\min \quad -\frac{1}{2}\left(x_1+1\right)^2+\frac{1}{2}x_2^2$$
$$\text{s.t.} \quad -x_1 \ge -1$$
$$x_1 \ge 0$$

Active constraint is $x_1 \le 1$

$\therefore A_+ = [-1 \quad 0]$

$x^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by inspection

Linear Inequality Constraints - II

$$\nabla f(x^*) = \begin{bmatrix} -\left(x_1^* + 1\right) \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\nabla^2 f(x^*) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Active constraint is $x_1 \leq 1$

$$\therefore \mathrm{A}_+ = \begin{bmatrix} -1 & 0 \end{bmatrix}$$

Linear Inequality Constraints - III

KKT Conditions

$Ax^* = b$

$\nabla f(x^*) = A^T\lambda$

$\lambda^* \geq 0$

$\lambda^*(Ax^* - b) = 0$

$$\lambda = \begin{bmatrix} ? \\ 0 \end{bmatrix} \quad \text{since second constraint is inactive}$$

$$\nabla f(x^*) = A^T\lambda \Rightarrow \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}\lambda \Rightarrow \lambda_1 = 2 \geq 0$$

$$\text{KKT Point: } x^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \lambda^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Linear Inequality Constraints - IV

Now look at SOSC

$$A_+ = [-1 \quad 0] \qquad Z_+ = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\nabla^2 f(x^*) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$Z_+' \nabla^2 f(x^*) \; Z_+ = [1]$ a p.d. matrix

Therefore SOSC are satistified.

X^* is a strict local minima

Why Necessary and Sufficient?

- Sufficient conditions are good for?
  - Way to confirm that a candidate point
  - is a minimum (local)
  - But…not every min satisfies any given SC
- Necessary tells you:
  - If necessary conditions don’t hold then you know you don’t have a minimum.
  - Under appropriate assumptions, every point that is a min satisfies the necessary cond.
  - Good stopping criteria
  - Algorithms look for points that satisfy Necessary conditions

You Try

- Solve the problem using above theorems: Verify x*=[8/9,4/18]' is optimal by checking KKT.
- Also check SOSC

$$
\begin{aligned}
\min \quad & 1/2x_1^2 + x_2^2 \\
s.t. \quad & 2x_1 + x_2 \geq 2 \\
& x_1 - x_2 \leq 1 \\
& x_1 \geq 0
\end{aligned}
$$

General Constraints

$$\begin{array}{lll} \min & f(x) & \\ s.t. & g_i(x) = 0 & i \in E \\ & g_i(x) \geq 0 & i \in I \end{array}$$

Active Set

- The active set at a point x consists of
  - Equality constraints
  - Inequality constraints at bound

$$A(x) = E \cup \{i \mid i \in I, g_i(x) = 0\}$$

Lagrangian Function

- Optimality conditions expressed using Lagrangian function

$$L(x,\lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x) = f(x) - \lambda' g(x)$$

and Jacobian matrix

$$\nabla g(x)'$$

were each row is a gradient of a constraint

Lagrangian Function

- Lagrangian Function

$$L(x,\lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x) = f(x) - \lambda' g(x)$$

- Lagragian Gradient

$$\nabla_x L(x,\lambda) = \nabla f(x) - \sum_{i=1}^{m} \lambda_i \nabla_x g_i(x)$$

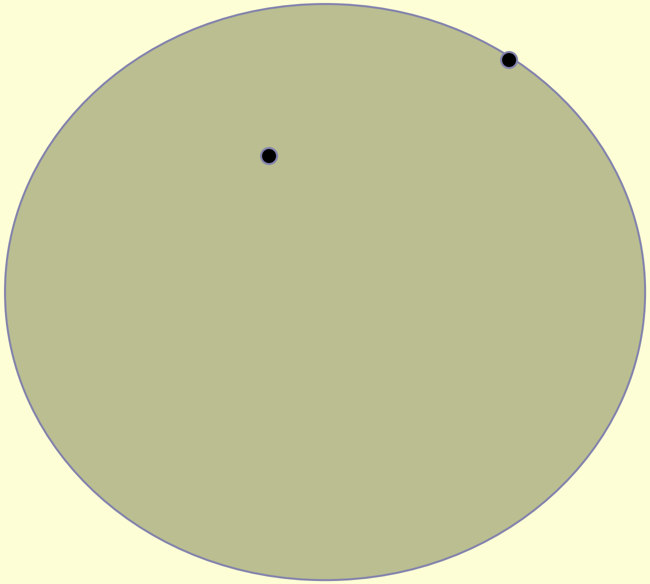
- Lagragian Hessian

$$\nabla^2_{xx} L(x,\lambda) = \nabla^2_{xx} f(x) - \sum_{i=1}^{m} \lambda_i \nabla^2_{xx} g_i(x)$$

Feasible descent directions

- A point is not optimal at x if we can take a small step s that is feasible and decrease the function

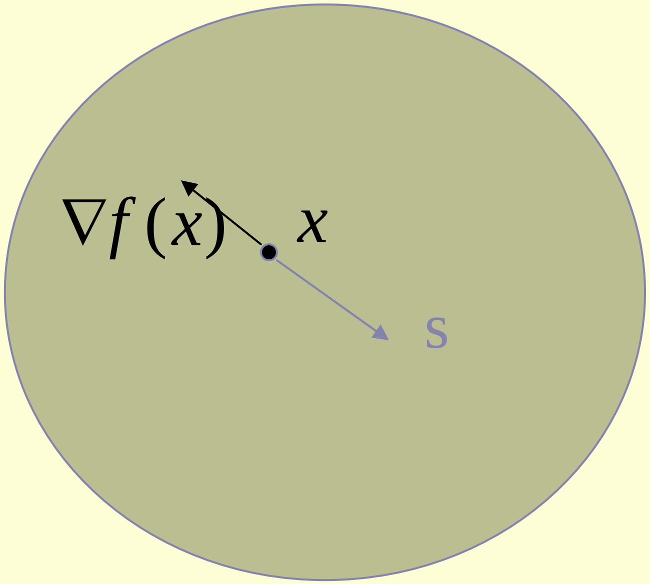
$$\min \quad f(x_1, x_2)$$
$$s.t. \quad 1 - x_1^2 - x_2^2 \geq 0$$



Feasible descent/Improvement directions

- Case 1: point in interior

$$\min \quad f(x_1, x_2)$$
$$s.t. \quad 1 - x_1^2 - x_2^2 \geq 0$$



Can only be optimal if

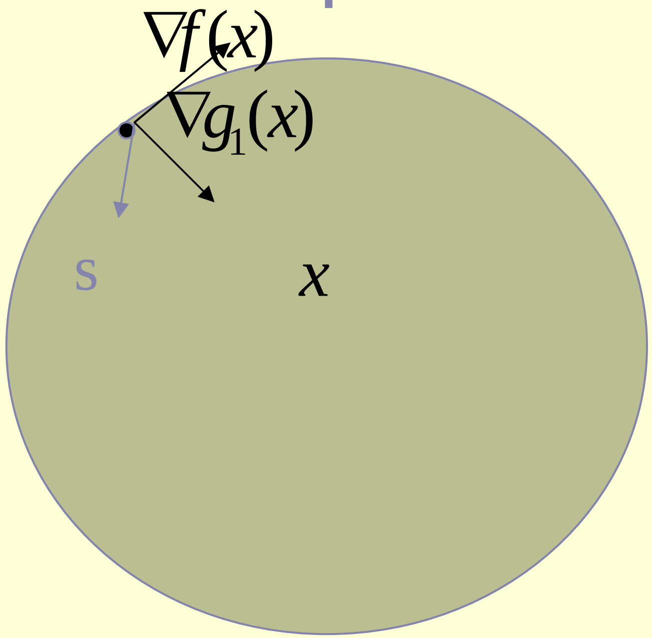
$$\nabla f(x) = 0$$

Feasible descent/Improvement directions

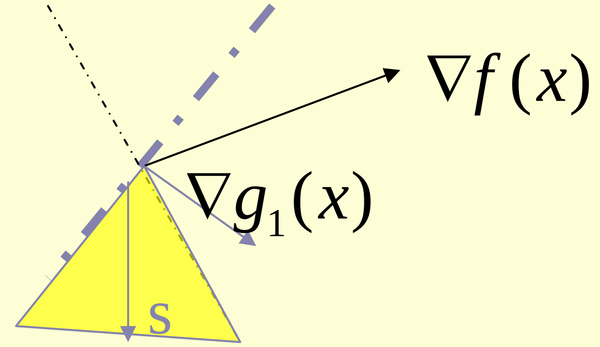
$$\min \quad f(x_1, x_2)$$

$$s.t. \quad 1 - x_1^2 - x_2^2 \geq 0$$

- Case 2: point at bound



Any point in cone is an Improvement directions



Can only be optimal if

$$\nabla f(x)'s < 0 \quad \nabla g_1(x)'s \geq 0$$

Has no solution

Feasible descent/Improvement directions

$$\min \quad f(x_1, x_2)$$
$$s.t. \quad 1 - x_1^2 - x_2^2 \geq 0$$

- This has no as solution s

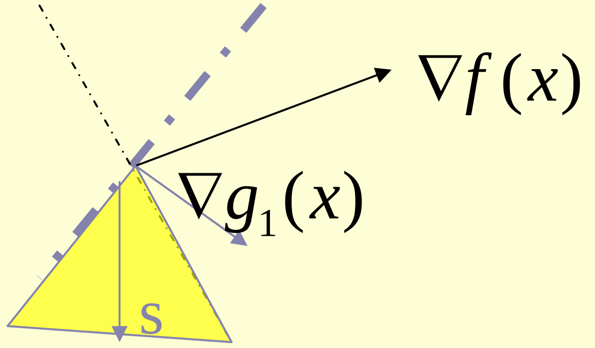
$$\nabla f\,(x^*)'s < 0 \quad \nabla g_1(x^*)'s \geq 0$$

- Same as saying there exists λ*

$$\nabla f\,(x^*) = \lambda^* \nabla g_1(x^*) \qquad \lambda^* \geq 0$$

- Or equivalently

$$\nabla L(x^*, \lambda^*) = 0 \qquad \lambda^* \geq 0$$



General Nonlinear Constraints

Want to solve
$$\min f(x)$$
$$s.t.\ g_1(x)=0\ g_2(x)\geq 0$$

Take FOTSE about a neighborhood of x to pick feasible descent directions at x assuming both constraints active

$$g_i(x+d)\approx g_i(x)+\nabla g_i(x)'d$$
$$f(x+d)\approx f(x)+\nabla f(x)'d$$

The set of $\nabla g_1(x)'d=0, \nabla g_2(x)'d\geq 0,\ and\ \nabla f(x)'d<0$ is the set of feasible descent directions for the linearized problem.

Rough idea

The set $\nabla g_1(x)'d = 0, \nabla g_2(x)'d \geq 0, \ \nabla f(x)'d < 0$

has no solution d, if and only if

$\nabla \mathrm{f(x)} = \nabla g_1(x)\lambda_1 + \nabla g_2(x)\lambda_2, \ \lambda_2 \geq 0$

or equivalently

$\nabla_x \mathrm{L}(\mathrm{x},\lambda_1,\lambda_2) = 0, \ \lambda_2 \geq 0$

has a solution

So same KKT as for linear case should work as long as “linearization is okay”. (waving hands)

Sufficient Conditions Nonlinear Equality

- If (x^*,λ^*) satisfies

$g(x^*) = 0$ Primal feasibility

$$\left.\begin{array}{l} \nabla f(x^*) = \nabla g(x^*)'\lambda^* \\ (\textit{equivalently}\, \nabla_x \mathrm{L}(\mathrm{x}^*,\lambda^*)=0) \end{array}\right\rangle \text{Dual feasibility}$$

and SOSC $Z'\nabla^2_{xx} L(x^*)Z$ *is p.d.*

Then x* is a strict local minimizer

Sufficient Conditions Nonlinear Equality

where Z is a basis matrix for Null(A) and A is Jacobian of active constraints

i.e.

For the jth row of A $\Rightarrow$

$g_j(x^*) = 0$ Active Constraint

$A_j = \nabla g_j(x^*)'$

Second Order Sufficient Conditions Nonlinear Inequality

- If (x^*,λ^*) satisfies

$g(x^*) \geq 0$ Primal feasibility

$$\left.\begin{aligned} &\nabla f(x^*) = \nabla g(x^*)'\lambda^* \\ &\quad (\textit{equivalently}\ \nabla_x \mathrm{L}(\mathrm{x}^*,\lambda^*)=0) \\ &\lambda^* \geq 0 \text{ (for inequalities only)} \end{aligned}\right\rangle \text{Dual feasibility}$$

$\lambda^{*\prime} g(x^*) = 0$ Complementarity

and SOSC $Z_+' \nabla^2_{xx} L(x^*) Z_+ \quad is\ p.d.$

Then x* is a strict local minimizer

Second Order Sufficient Conditions Nonlinear Inequality

where Z_+ is a basis matrix for Null(A_+) and A_+ corresponds to Jacobian of nondegenerate active constraints)

i.e. For the jth row of Jacobian $\Rightarrow$

$g_j(x^*) = 0$ Active Constraint

$\lambda_j^* > 0$ Nondegenerate

$$(A_+)_j = \nabla g_j(x^*)'$$

Sufficient Example

- Find solution and verify SOSC

$$\begin{aligned} \min \quad & -1/2(x_1+1)^2 - 1/2x_2^2 \\ s.t. \quad & 1/2x_1^2 + 1/2x_1^2 \le 1/2 \end{aligned}$$

$$x^* = [1,0]' \qquad \lambda^* = 2$$

Nonlinear Inequality Constraints - I

$$\min -\frac{1}{2}(x_1+1)^2 - \frac{1}{2}x_2^{\ 2}$$

$$\text{s.t.}\quad \frac{1}{2}x_1^{\ 2} + \frac{1}{2}x_2^{\ 2} \le \frac{1}{2}$$

$$\text{Guess}\ \ \mathrm{x}^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\nabla f(x) = \begin{bmatrix} -(x_1+1) \\ -x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\nabla^2 f(x^*) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Nonlinear Inequality Constraints - II

Has one active constraint: $\frac{1}{2}x_1^{\,2} + \frac{1}{2}x_2^{\,2} \leq \frac{1}{2}$

Jacobian: $\nabla \mathrm{g(x)}^{\mathrm{T}} = \begin{bmatrix} \mathrm{x}_1 & x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \end{bmatrix}$

$$L(x,\lambda) = f(x) + \lambda' g(x)$$

$$\nabla_x L(x,\lambda) = \nabla f(x) - \lambda_1 \nabla g_1(x)$$

$$= \begin{bmatrix} -2 \\ 0 \end{bmatrix} - \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Nonlinear Inequality Constraints - III

$$L(x,\lambda) = f(x) - \lambda' g(x)$$

$$\nabla_{xx}{}^2 L(x,\lambda) = \nabla^2 f(x) - \sum_i \lambda_i \nabla^2 g_i(x)$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - 2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \leftarrow \text{ positive definite}$$

$$\mathrm{Z}_+ = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ since } \nabla \mathrm{g(x)}^{\mathrm{T}} = \begin{bmatrix} -1 & 0 \end{bmatrix}$$

$$Z_+{}^T \nabla_{xx}{}^2 L(x,\lambda) Z_+ = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \leftarrow \text{positive definite}$$

so SOSC satisifed, x* is a strict local minimum

Sufficient Example

- Find solution and verify SOSC

$$\begin{array}{ll} \min & x_2 \\ s.t. & x_1^2 - x_2 = 0 \end{array}$$

$$x^* = [0,0]' \qquad \lambda^* = -1$$

Nonlinear Inequality Constraints - V

$$\min x_2$$

$$\text{s.t. } x_1^2 - x_2 = 0$$

$$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ by observation}$$

$$L(x,\lambda) = f(x) - \lambda' g(x)$$

$$= x_2 - \lambda(x_1^2 - x_2)$$

$$\nabla_x L(x,\lambda) = \nabla f(x) - \lambda' \nabla g(x)$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \lambda \begin{bmatrix} 2x_1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\nabla_x L(x^*,\lambda^*) = 0 \Rightarrow \quad = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \lambda^* \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0 \Rightarrow \lambda^* = -1$$

Nonlinear Inequality Constraints - VI

$$\nabla g(x)^T = \begin{bmatrix} 0 & -1 \end{bmatrix} \quad \therefore \quad Z_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \nabla_{xx}{}^2 L(x,\lambda) &= \nabla^2 f(x) - \sum_i \lambda_i \nabla^2 g_i(x) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - (-1)\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$Z_+{}^T \nabla_{xx}{}^2 L(x,\lambda) Z_+ = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \leftarrow \text{positive definite}$$

So SOSC are satisfied, and x* is a strict local minimum.

CAREFUL

- Sometimes things aren't nice and the KKT conditions don't exist even when a point is a min.
- We must impose additional conditions (constraint qualifications) to make sure the KKT conditions exist.

Example with no KKT point

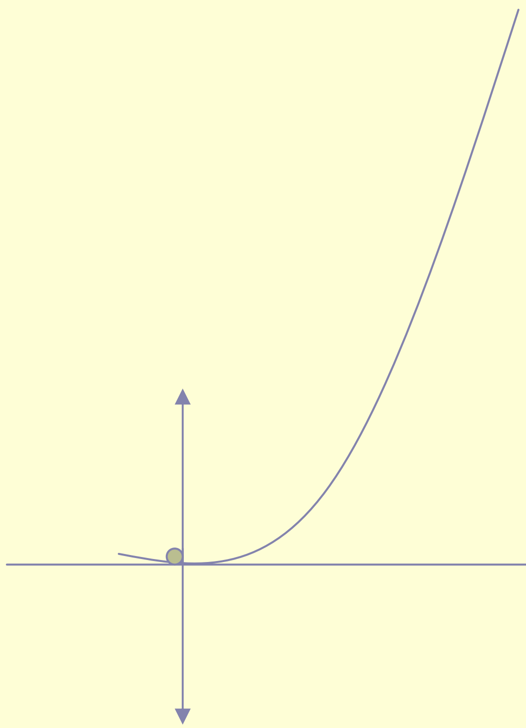
$$\min \; x$$

$$s.t. \quad x^2 - y \geq 0$$

$$y \geq 0$$

$$\nabla f(0,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\nabla g_1(0,0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \nabla g_2(0,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$\nabla f(0,0) + \lambda_1 \nabla g_1(0,0) + \lambda_2 \nabla g_1(0,0) \neq 0$$

$$\textit{for any } \lambda$$

What's wrong

Set of linearized feasible directions do not match the real set of feasible directions that can be achieved in the limit (called the tangent cone).

(Take NLP in fall to learn more about tangent cones)

Constraint Qualification

- Linear Independence CQ (LICQ)

  gradients of active constraints are linearly independent (also called regular point)
- Constraints are linear
- Convex program with strictly interior feasible points

…

More exists that make sure linearization works out okay.

First Order Necessary Conditions- Equality

- If x* is a local min of f over {x|g(x)=0}, and LICQ (or other CQ) then

$$\left.\begin{array}{l} \text{there exists } \lambda^* \\ \nabla_x L(x^*, \lambda^*) = 0 \\ g(x^*) = 0 \end{array}\right\rangle$$

Second Order Necessary Conditions- Equality

- If x* is a local min of f over {x|g(x)=0}, Z is a null-space matrix of the Jacobian ∇g(x*)', and LICQ then

$$\left.\begin{array}{l}\text{there exists } \lambda^* \\ \nabla_x L(x^*, \lambda^*) = 0 \text{ or equivalently } Z'\nabla f(x^*) = 0 \\ g(x^*) = 0\end{array}\right\rangle$$

$$and\ Z'\nabla^2_{xx} L(x^*)Z\ \ is\ \ p.s.d.$$

Necessary Conditions Inequality

- If x* is a local min of f over {x|g(x)>=0}, Z is a null-space matrix of the Jacobian ∇g(x*)’ of the active constraints, and x* is a regular point (LICQ) then

$$\left.\begin{array}{l}\text{there exists } \lambda^* \\ \nabla_x L(x^*, \lambda^*) = 0 \text{ or equivalently } Z'\nabla f(x^*) = 0 \\ g(x^*) \geq 0 \\ \lambda^{*'} g(x^*) = 0 \\ \textit{and } Z'\nabla^2_{xx} L(x^*) Z \textit{ is p.s.d.}\end{array}\right\rangle$$

LICQ=Regular point

- If x* is a regular point with respect to the constraints g(x*) if the gradient of the active constraints are linearly independent.
- For equality constraints, all constraints are active so

$$\nabla g_I(x^*)'$$

should have linearly independent rows.

Necessary Example

- Show optimal solution x*=[1,0]' is regular and find KKT point

$$
\begin{aligned}
\max \quad & x_1 \\
s.t. \quad & x_1^2 + x_2^2 \le 1 \\
& (x_1 - 1)^3 - x_2 \le 0
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & -x_1 \\
s.t. \quad & -(x_1^2 + x_2^2) + 1 \ge 0 \\
& -(x_1 - 1)^3 + x_2 \ge 0
\end{aligned}
$$

KKT point

$$x^* = [1, 0]'$$

$$\lambda^* = [1/2, 0]'$$

Constraint Qualifications

- Regularity or LICQ is an example of a constraint qualification CQ.
- The KKT conditions are based on linearizations of the constraints.
- CQ guarantees that this linearization is not getting us into trouble. Problem is

  KKT point might not exist.
- There are many other CQ,e.g., for convex inequalities Slater is there exists g(x)<0.
- Note CQ not needed for linear constraints.

Necessary Conditions General

- If x* satisfies LICQ and is a local min of f over {x|g(x)>=0,h(x)=0},

there exists $\lambda_I^* \geq 0, \lambda_E^*$

$$0 = \nabla_x L(x^*, \lambda_I^*, \lambda_E^*) = f(x^*) + \sum_i \lambda_i^* g_i(x^*) + \sum_i \lambda_i^* h_i(x^*)$$

$$g(x^*) \geq 0, h(x^*) = 0$$

$$\lambda^{*'} g(x^*) = 0$$

$and\ Z' \nabla_{xx}^2 L(x^*) Z\ is\ p.s.d$

$where$ Z is the null space matrix for $A(x^*)$ (active constraints)

General SOSC

Consider problem min of f over {x|g(x)>=0,h(x)=0},

If there exists x*, $\lambda_I^* \geq 0, \lambda_E^*$ such that

$$0 = \nabla_x L(x^*, \lambda_I^*, \lambda_E^*) = f(x^*) + \sum_i \lambda_i^* g_i(x^*) + \sum_i \lambda_i^* h_i(x^*)$$

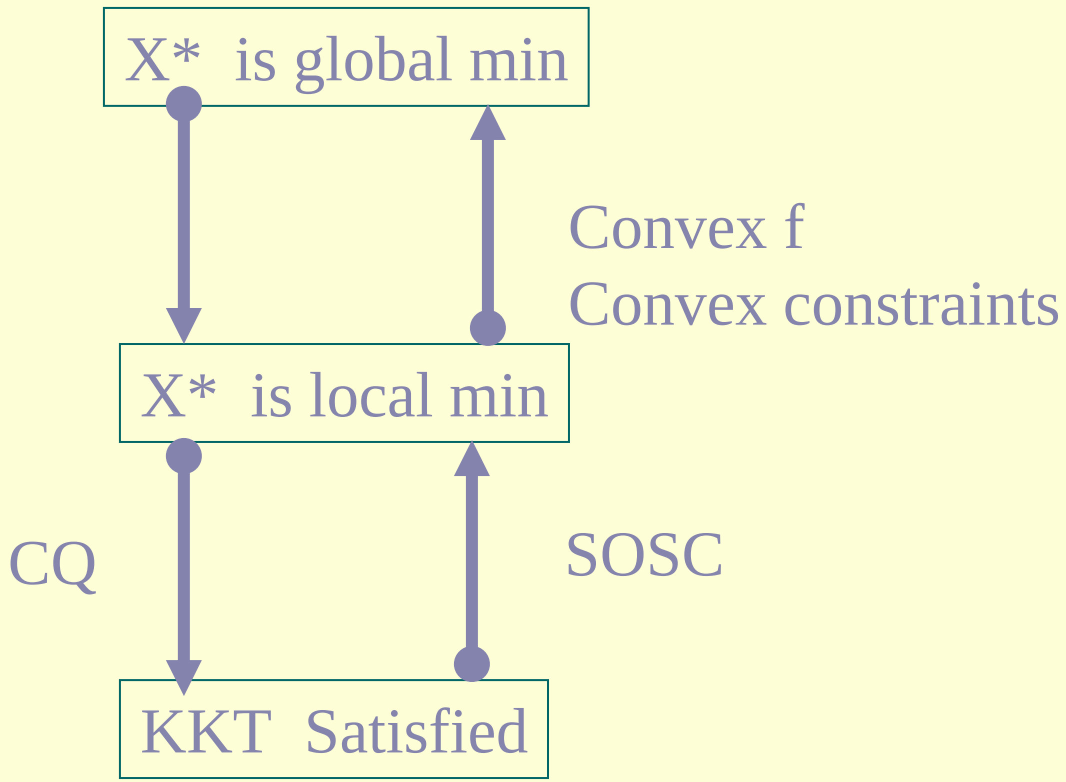
$$g(x^*) \geq 0, h(x^*) = 0$$

$$\lambda^{*'} g(x^*) = 0 \quad \text{and } Z_+' \nabla_{xx}^2 L(x^*) Z_+ \text{ is } p..d$$

where Z_+ is the null space matrix for $A_+(x^*)$ (active constraints with positive multipliers)

then x* is a strict local minimizer

KKT Summary


X* is global min
Convex f
Convex constraints
X* is local min
CQ
SOSC
KKT Satisfied