Computational Optimization

Constrained Optimization

Algorithms – Feasible Descent Methods Continued

Next Problem

- Consider Next Hardest Problem

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & Ax \geq b \end{aligned}$$

- How could we adapt gradient projection or other linear equality constrained techniques to this problem?

Consider the following problem

- Original standard form

$$\begin{aligned} \min \quad & 2x_1^2 + 2x_2^2 - 2x_1x_2 - 4x_1 - 6x_2 \\ s.t. \quad & x_1 + x_2 \le 2 \\ & x_1 + 5x_2 \le 5 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \min \quad & 2x_1^2 + 2x_2^2 - 2x_1x_2 - 4x_1 - 6x_2 \\ s.t. \quad & -x_1 - x_2 \ge -2 \\ & -x_1 - 5x_2 \ge -5 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{aligned}$$

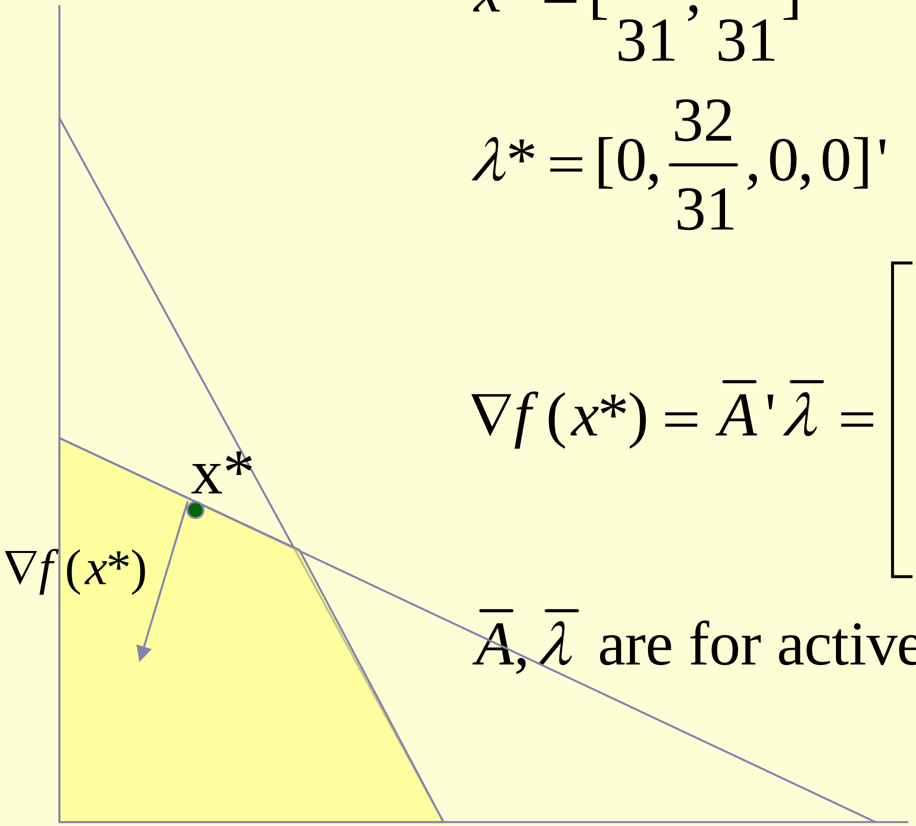
KKT Point (you check)

$$x^* = [\frac{35}{31}, \frac{24}{31}]'$$

$$\lambda^* = [0, \frac{32}{31}, 0, 0]'$$

$$\nabla f(x^*) = \bar{A}'\bar{\lambda} = \begin{bmatrix} -\frac{32}{31} \\ -\frac{160}{31} \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \end{bmatrix} \frac{32}{31}$$

$\bar{A}, \bar{\lambda}$ are for active constraints only



KKT conditions same as equality KKT for active constraints

Key is Optimality Conditions

- Optimality conditions for inequality are the same as the ones for equality using only the active strengths provided multipliers are nonnegative.
- Basic Strategy:

  Guess which constraints are active.

  Called the Working Set *W*.

What if W is wrong?

$x_1 = [0,0]'$ $W = \{3,4\}$ since last two constraints active

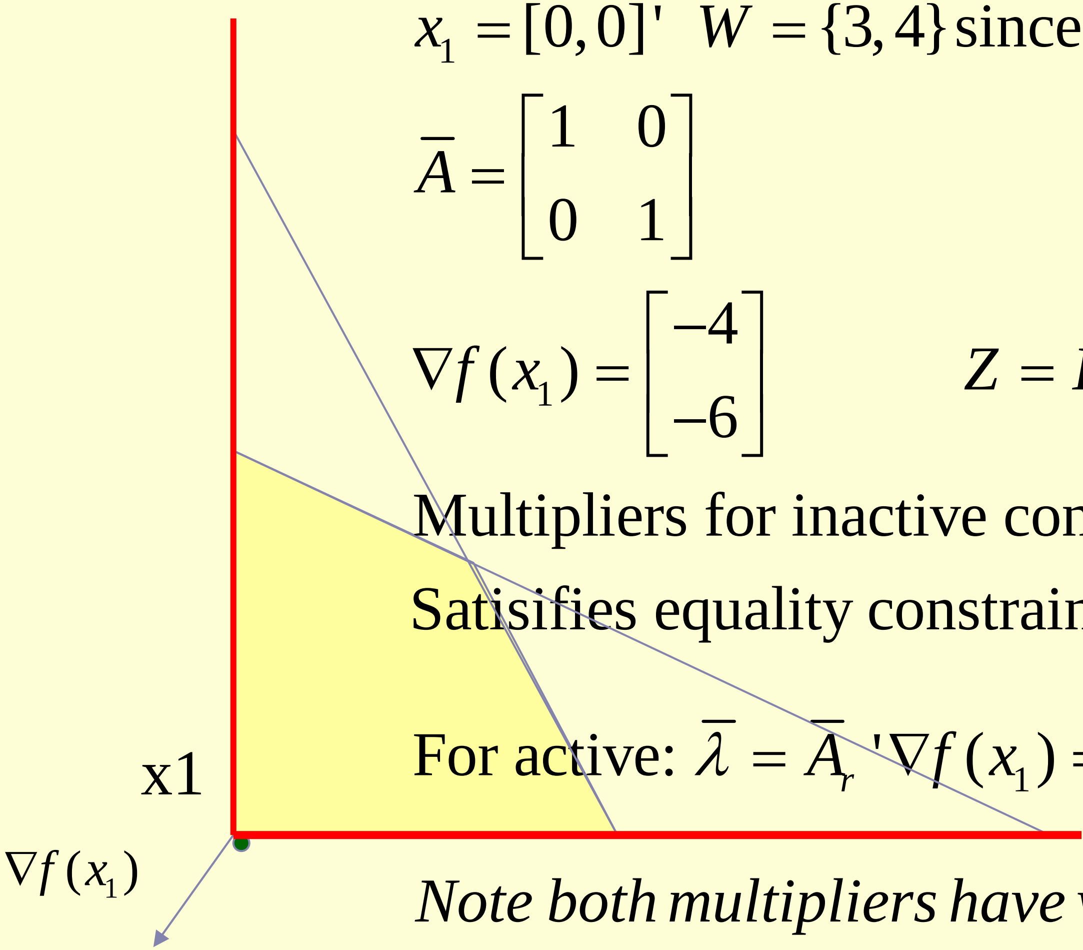
$$\bar{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\nabla f(x_1) = \begin{bmatrix} -4 \\ -6 \end{bmatrix} \qquad Z = I - \bar{A}'(\bar{A}\bar{A}')^{-1}\bar{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Multipliers for inactive const. are 0

Satisifies equality constrained part $Z'\nabla f(x_1) = 0$

For active: $\bar{\lambda} = \bar{A}_r{}'\nabla f(x_1) = \left(\bar{A}'(\bar{A}\bar{A}')^{-1}\right)'\nabla f(x_1) = \begin{bmatrix} -4 \\ -6 \end{bmatrix}$

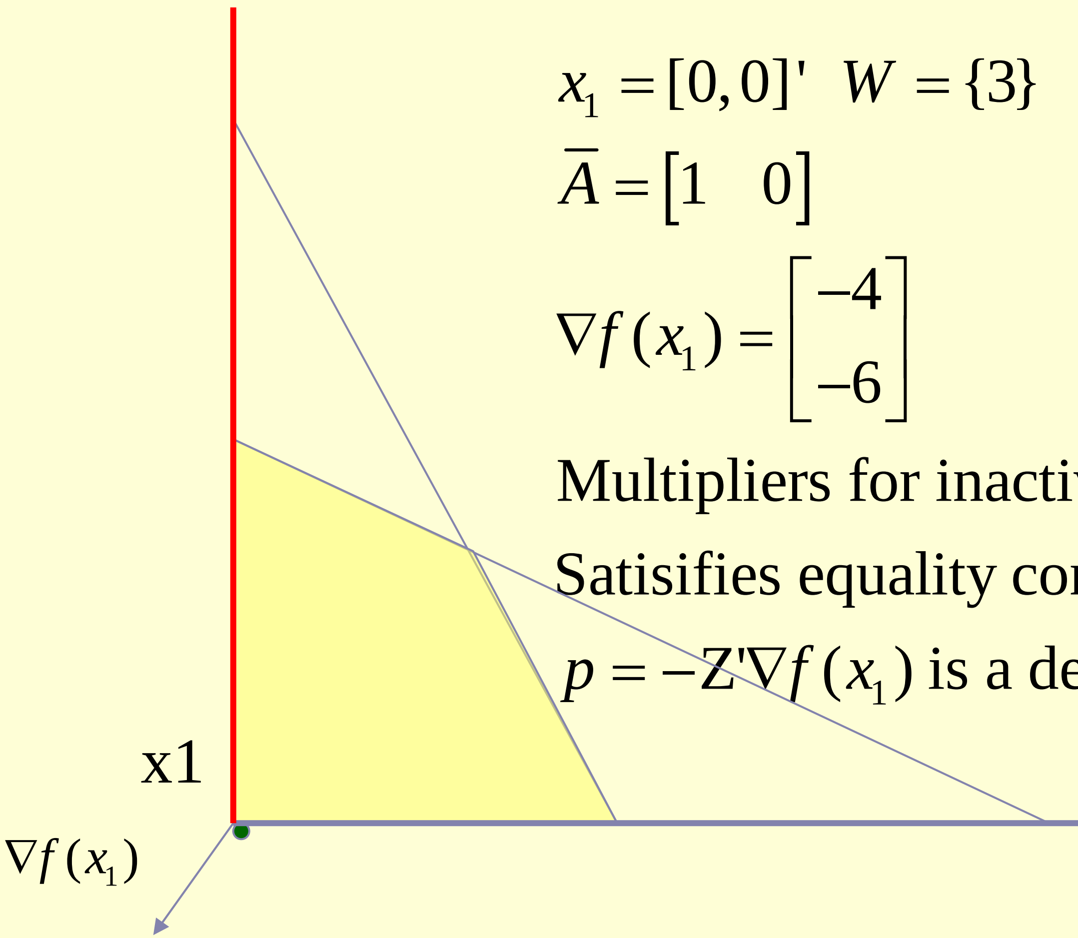


Note both multipliers have wrong sign

Case I: Optimal for W but not for original problem

- If problem is optimal with respect to working set. Construct multipliers.
- If any multiplier is negative, drop most negative multiplier.

Try again



$$x_1 = [0,0]' \quad W = \{3\}$$

$$\bar{A} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\nabla f(x_1) = \begin{bmatrix} -4 \\ -6 \end{bmatrix} \qquad Z = I - \bar{A}'(\bar{A}\bar{A}')^{-1}\bar{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Multipliers for inactive const. are 0

Satisifies equality constrained part $Z'\nabla f(x_1) \neq 0$

$p = -Z'\nabla f(x_1)$ is a descent direction!

How far can you step?

$$x_1 = [0,0]' \quad W = \{3\}$$

$$\bar{A} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

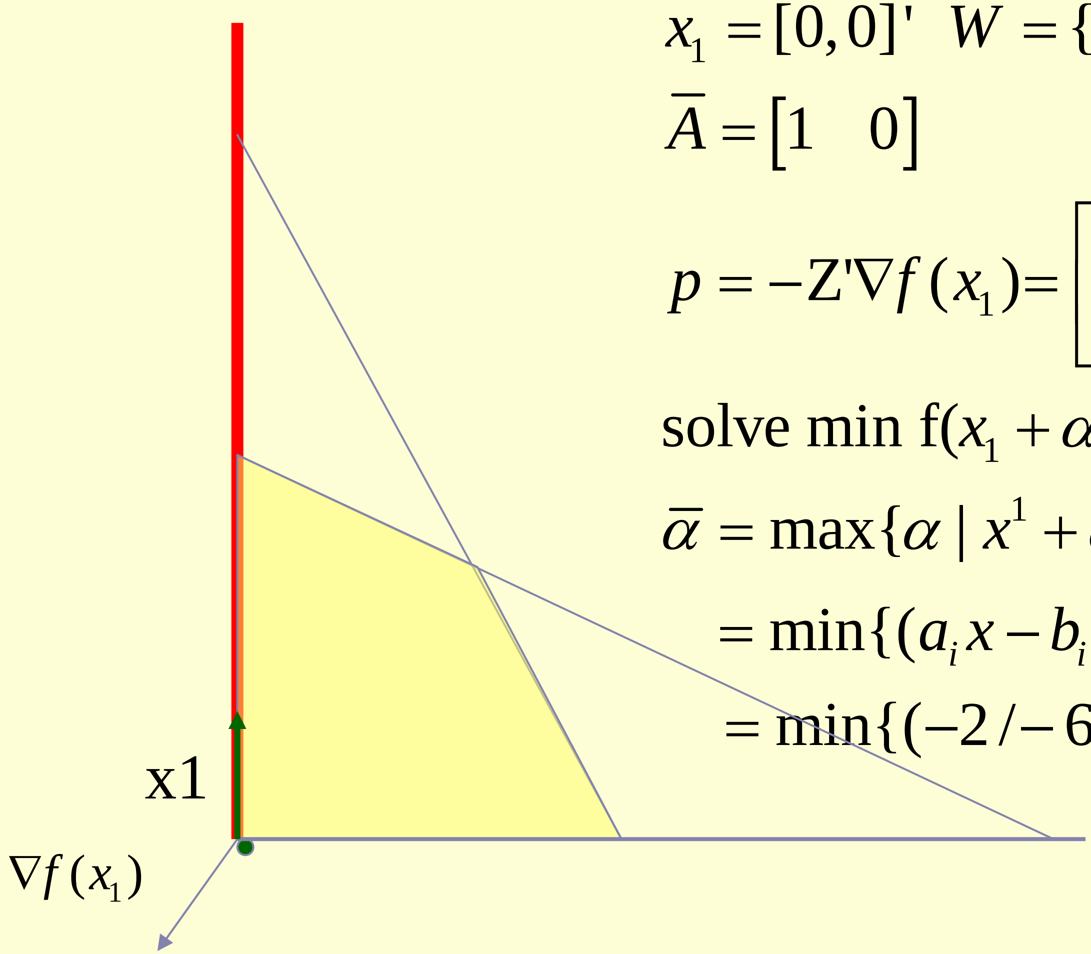
$$p = -Z'\nabla f(x_1) = \begin{bmatrix} 0 \\ 6 \end{bmatrix} \text{ is a descent direction}$$

$$\text{solve min f}(x_1 + \alpha p)\ s.t.\ 0 \le \alpha \le \bar{\alpha}$$

$$\bar{\alpha} = \max\{\alpha \mid x^1 + \alpha p \ feasible\}$$

$$= \min\{(a_i x - b_i)/(-a_i^t p) : a_i^T p < 0, i \notin W\}$$

$$= \min\{(-2/-6, -5/-30\} = 1/6$$



Move along constraint

$x_1 = [0, 0]' \quad W = \{3\}$

$\bar{A} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$p = -Z'\nabla f(x_1) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$ is a descent direction

solve min f$(x_1 + \alpha p)$ $s.t.\ 0 \le \alpha \le \bar{\alpha}$

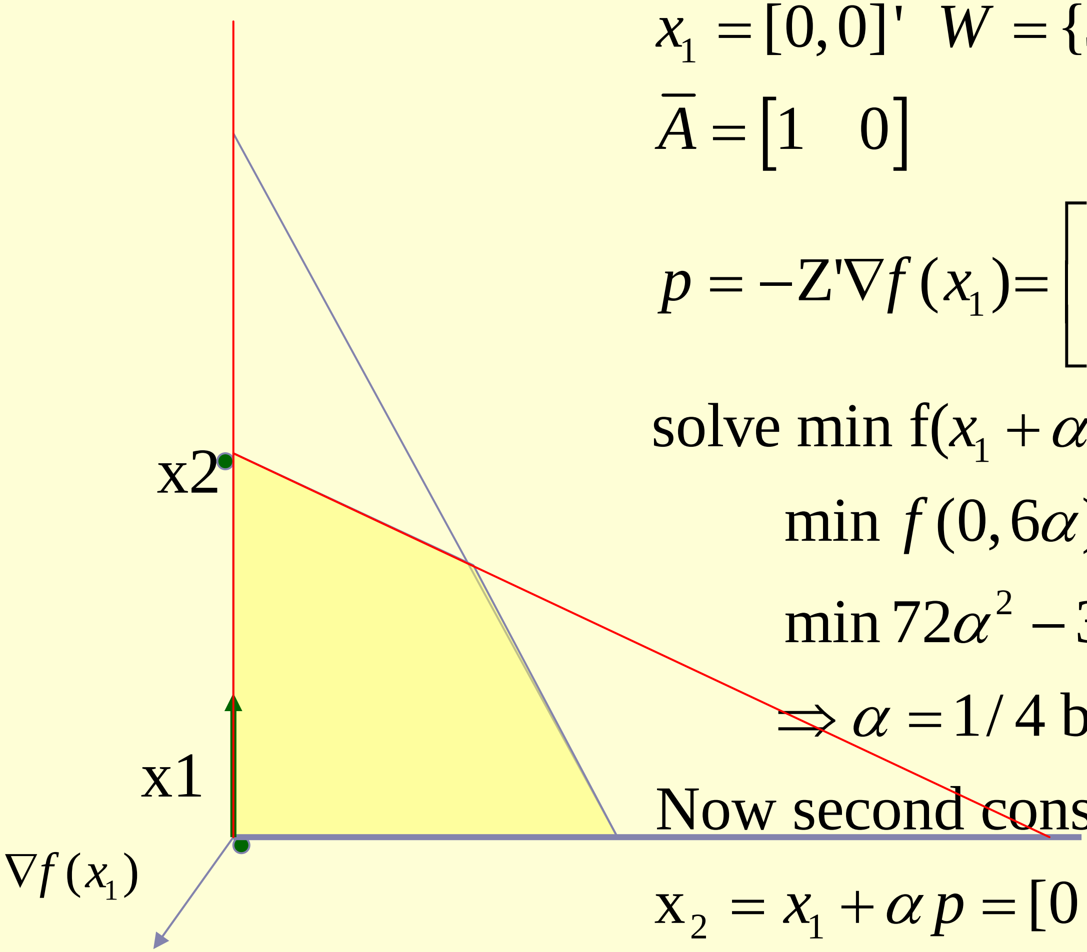
$\min f(0, 6\alpha)\ s.t.\ 0 \le \alpha \le 1/4$

$\min 72\alpha^2 - 36\alpha$

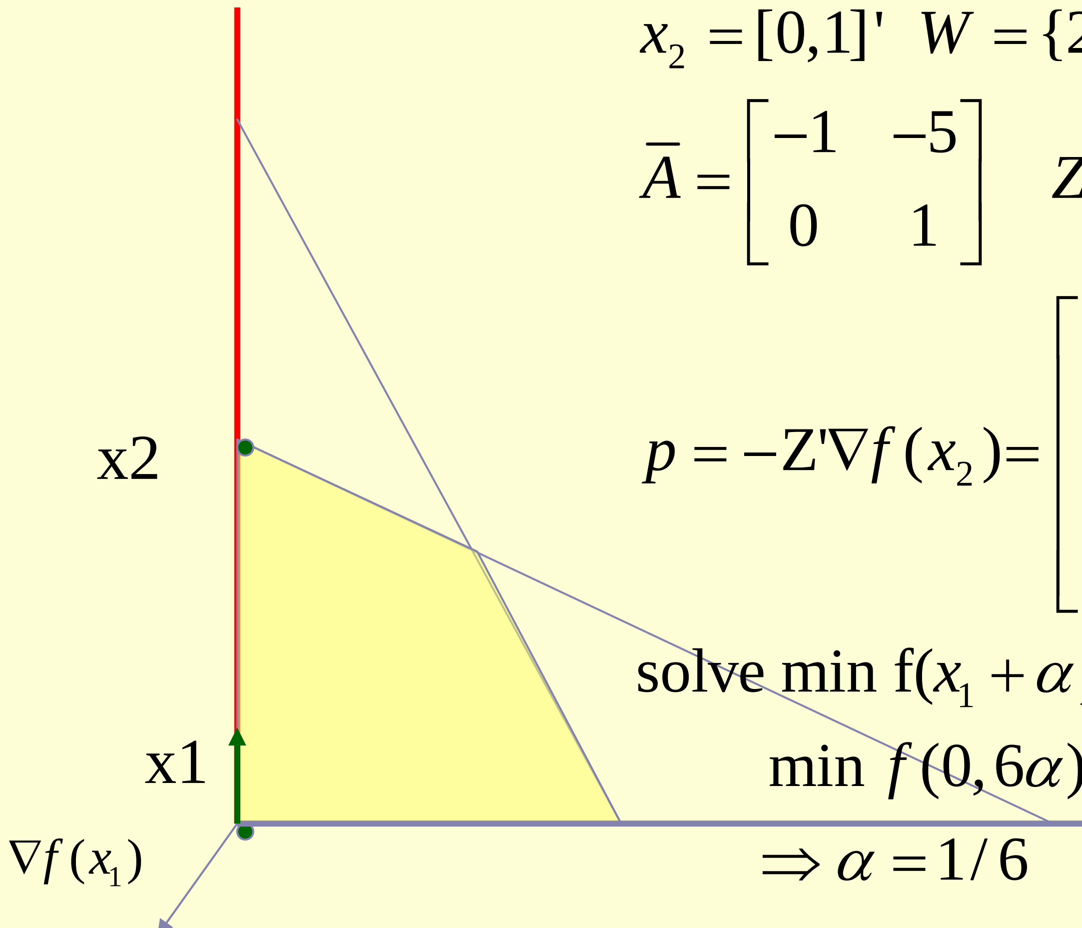
$\Rightarrow \alpha = 1/4$ but too large so take $\alpha = \bar{\alpha}$

Now second constraint is active too.

$x_2 = x_1 + \alpha p = [0\ 1]'$



Add New Constraint



$$x_2 = [0,1]' \quad W = \{2,3\}, \nabla f(x_2) = [-6-2]'$$

$$\bar{A} = \begin{bmatrix} -1 & -5 \\ 0 & 1 \end{bmatrix} \quad Z = \begin{bmatrix} 25 & -5 \\ -5 & 1 \end{bmatrix} / 26$$

$$p = -Z'\nabla f(x_2) = \begin{bmatrix} \frac{70}{13} \\ -\frac{14}{13} \end{bmatrix} \propto \begin{bmatrix} \quad \end{bmatrix} \text{is a descent direction}$$

$$\text{solve min f}(x_1 + \alpha p)\; s.t.\; 0 \le \alpha \le \bar{\alpha}$$

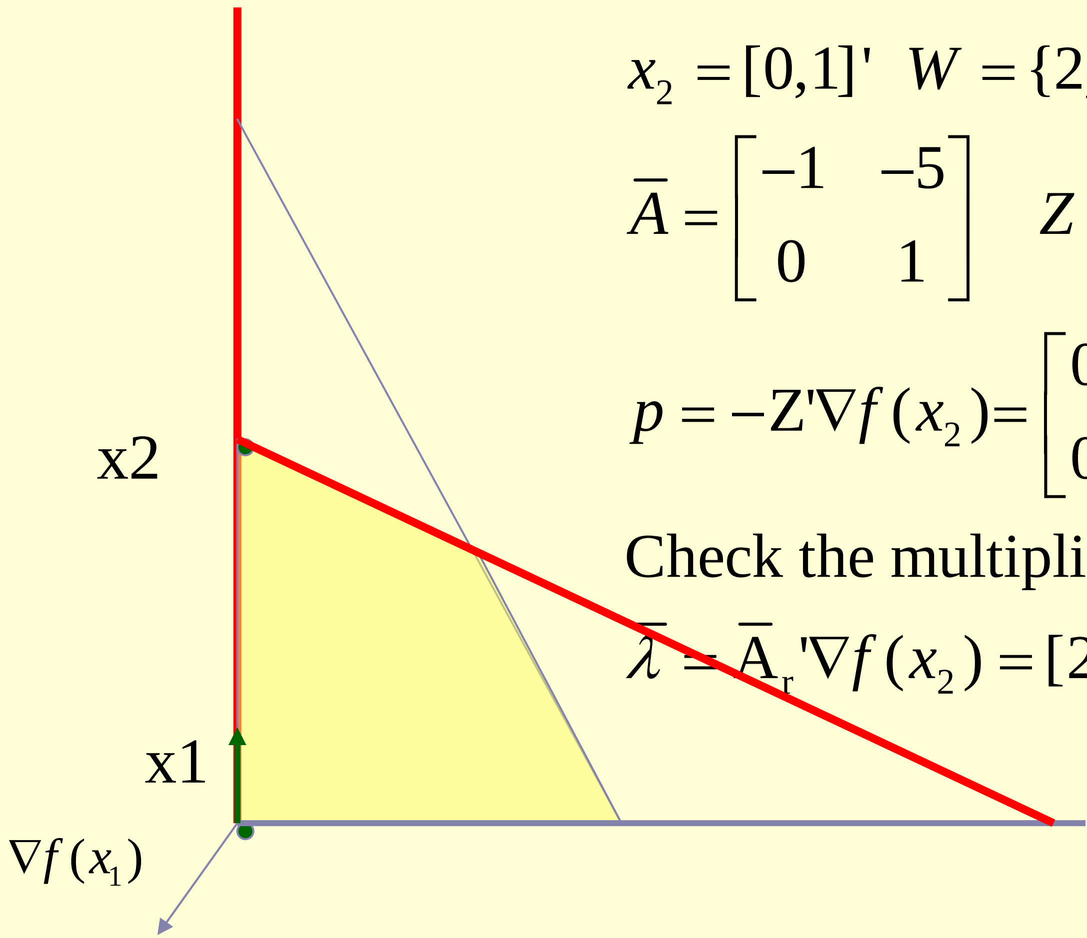
$$\min f(0, 6\alpha)\; s.t.\; 0 \le \alpha \le 1/4$$

$$\Rightarrow \alpha = 1/6$$

Now third constraint is active.

$$x_2 = x_1 + \alpha p = [0\;1]'$$

Try new active set



$$x_2 = [0,1]' \quad W = \{2,3\}, \nabla f(x_2) = [-6 - 2]'$$

$$\bar{A} = \begin{bmatrix} -1 & -5 \\ 0 & 1 \end{bmatrix} \quad Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

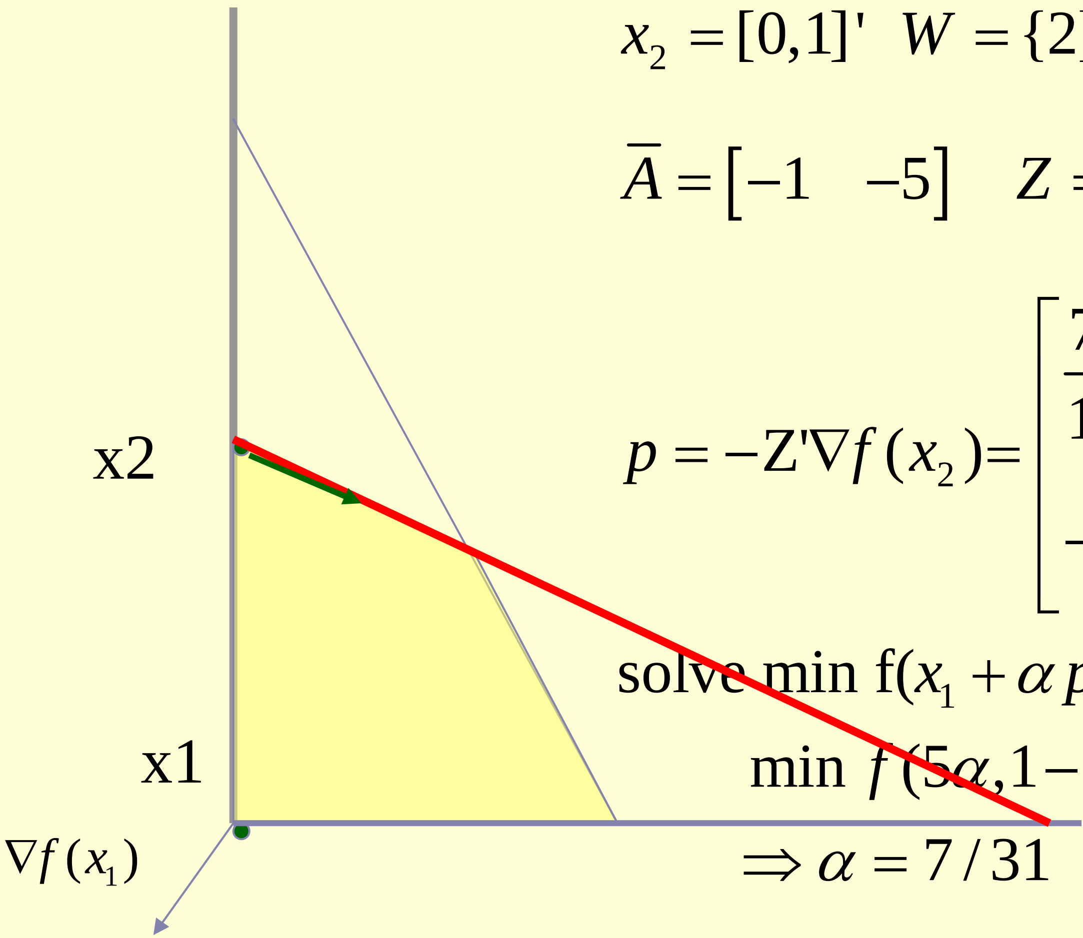
$$p = -Z'\nabla f(x_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ optimal with respect to active

Check the multipliers

$$\bar{\lambda} = \bar{A}_r'\nabla f(x_2) = [2/5 \quad -28/5]$$

Drop the third constraint

Drop and try again



$$x_2 = [0,1]' \quad W = \{2\}, \nabla f(x_2) = [-6 - 2]'$$

$$\bar{A} = \begin{bmatrix} -1 & -5 \end{bmatrix} \quad Z = \begin{bmatrix} 25 & -5 \\ -5 & 1 \end{bmatrix} / 26$$

$$p = -Z'\nabla f(x_2) = \begin{bmatrix} \frac{70}{13} \\ -\frac{14}{13} \end{bmatrix} \propto \begin{bmatrix} 5 \\ -1 \end{bmatrix} \text{is a descent direction}$$

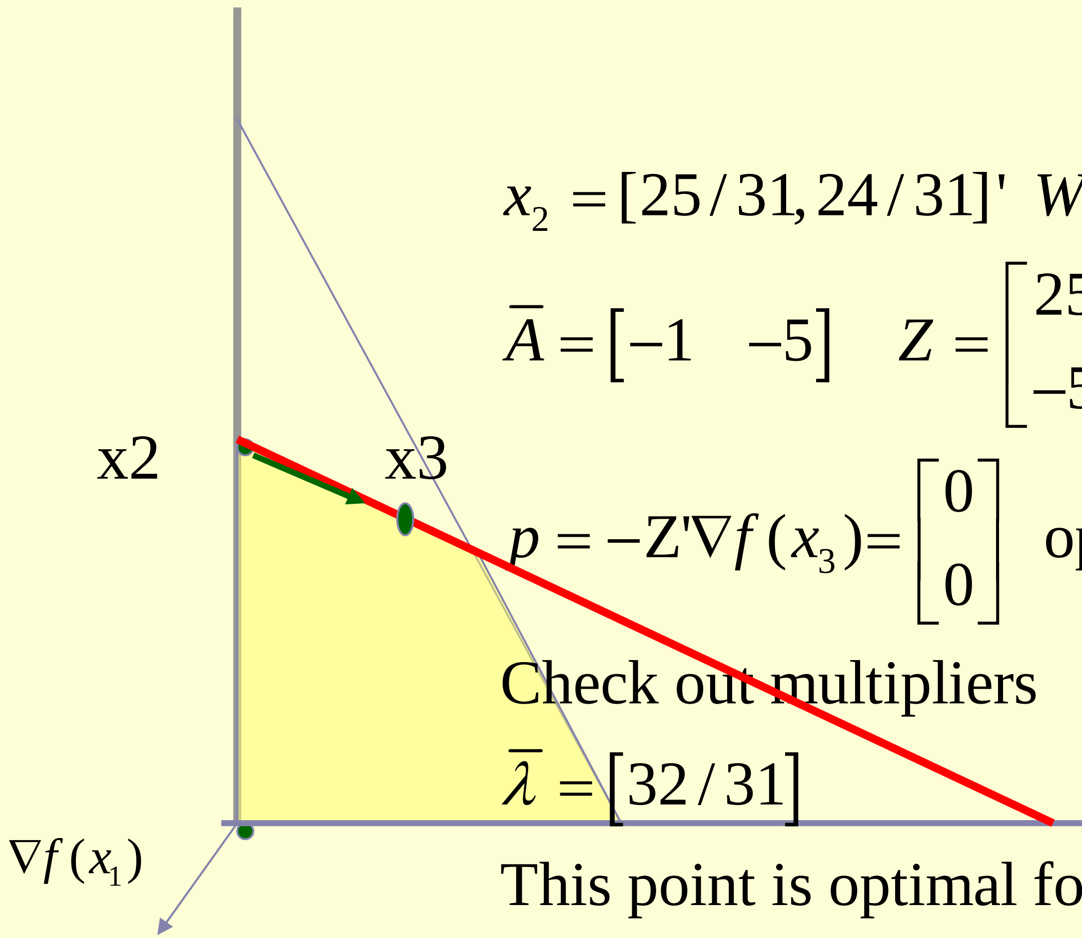
$$\text{solve min f}(x_1 + \alpha p)\ s.t.\ 0 \le \alpha \le \bar{\alpha}$$

$$\min f(5\alpha, 1-\alpha)\ s.t.\ 0 \le \alpha \le 1/4$$

$$\Rightarrow \alpha = 7/31$$

$$x3 = x2 + \alpha p = [25/31, 24/31]'$$

Check out X3



$$x_2 = [25/31, 24/31]' \quad W = \{2\}, \nabla f(x_2) = [-32/31 \quad -160/$$

$$\bar{A} = \begin{bmatrix} -1 & -5 \end{bmatrix} \quad Z = \begin{bmatrix} 25 & -5 \\ -5 & 1 \end{bmatrix} / 26$$

$$p = -Z'\nabla f(x_3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ optimal for equality constrained

Check out multipliers

$$\bar{\lambda} = \begin{bmatrix} 32/31 \end{bmatrix}$$

This point is optimal for the whole problem

Active Set Method pg 504 NS

1. Let $W = \varnothing \quad \bar{A} = 0 \;\; \bar{\lambda} = 0$
2. If $Z'\nabla f(x_k) = 0 \text{ or equivalently } \nabla f(x_k) = \bar{A}'\bar{\lambda}$ then:
3. If W=∅ then unconstrained min
4. Else compute $\bar{\lambda} = \bar{A}_r'\nabla f(x_k)$
5. If $\bar{\lambda} \geq 0$, stop then optimal.
   otherwise drop constraint with most negative multiplier
6. Compute feasible descent direction p with respect to W

Active Set Method

7. Compute step satisfying:

$$f(x_k + \alpha p) < f(x_k) \; and \; \alpha \leq \bar{\alpha} \; (\max \; feasible \, step)$$

8. Update: $x_{k+1} = x_k + \alpha p$
9. If hit bound, add one constraint to working set and update $W, \bar{A}, Z, \bar{A}_R$
10. k=k+1
11. Goto 2

Active Set applied to QP

- Section 16.5 in NW
- QP

$$\min \frac{1}{2} x'Qx + x'c$$

$$s.t. \quad a_i^t x \geq b_i \qquad i = 1..m$$

or if we new active set A(x*)

$$\min \frac{1}{2} x'Qx + x'c$$

$$s.t. \quad a_i^t x = b_i \qquad i \in A(x^*)$$

Feasible Descent Problem 16.39

We are at xk with gradient gk=Gxk+c

Next set is x = xk+p

$$\min \frac{1}{2}(x+p_k)'Q(x+p_k)+(x+p_k)'c$$

$$s.t. \quad a_i^t(x+p_k)=b_i \;\; i\in W_k$$

Subproblem has nice structure

- Equivalent problem

$$\min \frac{1}{2}(p_k)'Q(p_k) + (p_k)'c + (\text{constant})$$

$$s.t. \quad a_i^t p_k = 0_i \quad i \in W_k$$

- Solution has closed form:

$$\begin{bmatrix} G & \bar{A}' \\ \bar{A} & 0 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} = 0$$

Algorithm 16.3 Active set for Convex QP

Start with feasible x0

W0 = A(x0)

For k = 0,1 ,2,….

Solve active set QP(16.43) for pk

if pk = 0

Compute Lagragian multipliers λ

if λ≥0 then stop xk optimal

else

$$j \leftarrow \arg\min_{j \in Wk \cap I} \lambda_j$$

$$x_{k+1} \leftarrow x_k ; W_{k+1} \leftarrow W_k \setminus j$$

else (pk not 0)

Compute stepsize αk from 16.41

$$x_{k+1} \leftarrow x_k + \alpha_k p_k$$

if there are blocking constraints

obtain W_{k+1} by adding one of blocking constraints to W_k

else

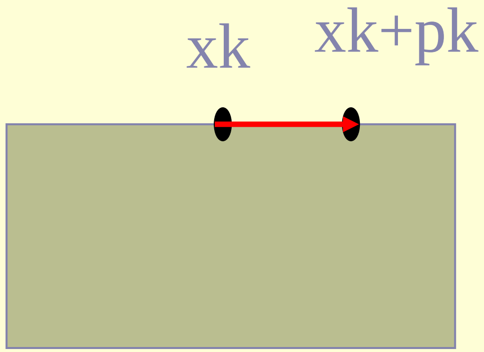
$$W_{k+1} \leftarrow W_k$$

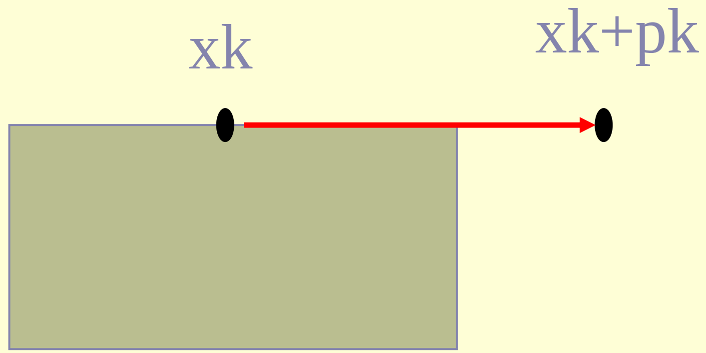
end

Stepsize?

- Case 1: Optimal step x=xk+pk is feaible
- Case 2: Optimal step x=xk+pk is not feasible. Step until hit bound.

$$\alpha_k = \min\left(1, \min_{i \notin W_k, a_i' p^k < 0} \right)$$





Let's try

- Try this problem using active set method

$$\begin{aligned} \min \quad & x_1^2 + x_1 x_2 + 2x_2^2 - 6x_1 - 14x_2 \\ s.t. \quad & x_1 + x_2 \le 2 \\ & x_1 - 2x_2 \ge -3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

- Starting point x =[0 0]'
- Optimal point x*=[1 5]'/3

Variations

- Different ways of constructing Z produces different algorithms

  For example A= [B N] variable reduction produces simplex method.
- Can speed up by solving subproblems inexactly.
- Note still just a gradient descent method at heart. Can zigzag.

QR Factorization Very Good

Solve subproblem

$$Gx = \bar{A}'\lambda$$

$$\bar{A}'x = b$$

$$\bar{A}' = QR = [Q_1 \quad Q_2]\begin{bmatrix} R_1 \\ 0 \end{bmatrix} \quad A = [Q_1 \; Q_2]\begin{bmatrix} R \\ 0 \end{bmatrix}$$

$$\text{where } A \in m \times n, Q_1 \in n \times m, Q_2 \in n \times (n-m), R_1 \in m \times m$$

$$Z = Q_2 \qquad A_r = Q_1 R_1^{-T}$$

Using QR factorization

$$x = Q_1 R_1^{-T} b, \qquad \lambda = Q_1 R_1^{-T} Q x_k$$

Update QR when adding/delete column

- Add constraint to working set

$$A_k' = [A_k \; a_j]$$

- Delete constraint from working set
- QR can be efficiently updated

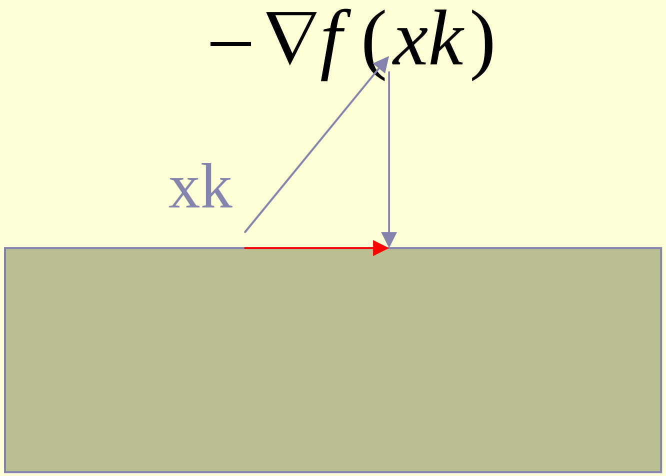
  see text, matlab commands

  qr,qrinsert,qrdelete

Gradient Projection Method

For equality constrained approach we projected the gradient back to the feasible region.

We could do this for Inequalities too if projection is cheap



$$\min \frac{1}{2} x'Qx + x'c$$

$$s.t. \quad l \le x \le u$$

Projection for bounds constraints

Projection is closest point in the set to x

$$\min_s \frac{1}{2} \| x - s \|^2 \;\; s.t. \;\; \ell \leq s \leq u$$

$$s_i = \begin{cases} \ell_i & if \quad x_i < \ell_i \\ x_i & if \quad \ell_i \leq x_i \leq u_i \; x_i \\ u_i & if \quad x_i > u_i \end{cases}$$

Cauchy Point

- Take a step $x_i - t g_i$
- The projected step is a function of t

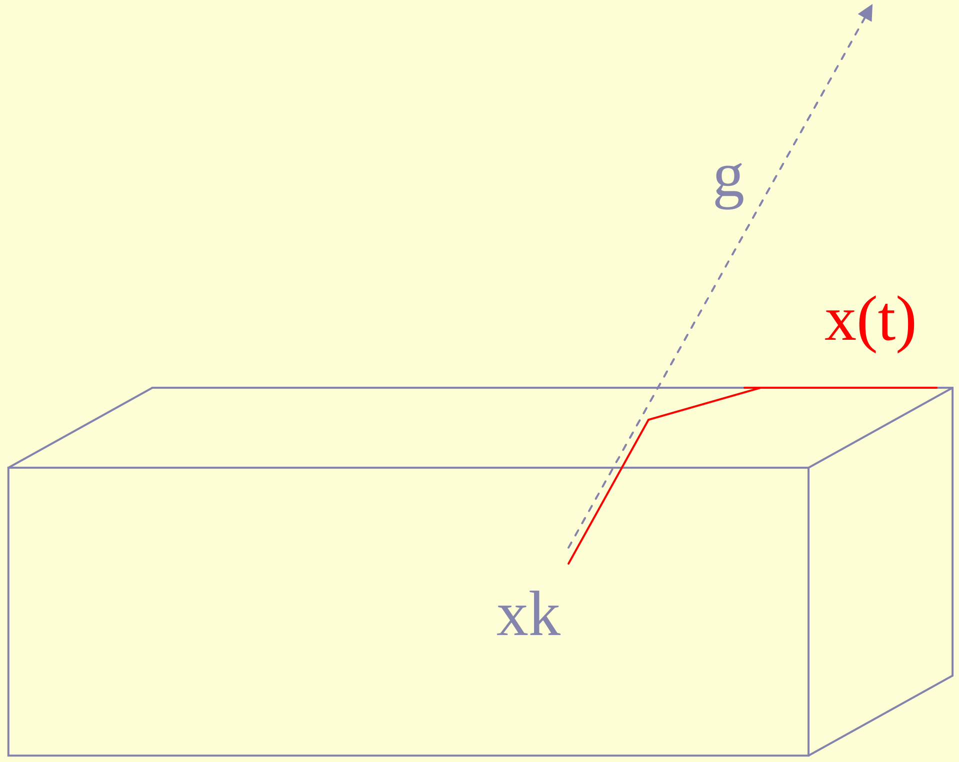
$$x(t) = P(x_i - t g_i)$$

- Cauchy point finds t that minimizes

$$q(x(t))$$

e.g. An exact linesearch along the projected direction

Plus: Cauchy point can change many constraints in working set



Gradient Projection Method for QP NW 16.5

Start with feasible x0

For x = 0,1,2,…

if check KKT to see if optimal

set x =xk and find cauchy point xc;

x_{k+1} is an approximate solution of QP using active set of xc fixed and rest feasible

approximation just needs to find feasible decrease.

End;

Active Set Summary

- Active set QP methods widely used
- Simplex method is an active set method for LP
- Can do hot start (start from good solution)
- Projection method effective when constraints have nice projections.
- Gradient methods can still be slow
- Interior methods usually better for LP and QP but no hot start.