A Link Based Quasi-Variational Inequality Model for Dynamic User Equilibria, towards Real Time Traffic Operations

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***Abstract*—We propose a link based quasi-variational inequality formulation for the deterministic dynamic user equilibrium problem with single-user-class and fixed demands. The solution existence and boundedness condition for the proposed model is established using a newly developed theory for a group of quasi-variational inequality problems. The model can be solved by the damped Newton approach based on two observations: a) the sub-problem can be reformulated as a nonlinear complementarity problem which can be solved very efficiently by existing solution techniques, and b) the step size of the algorithm can be calculated using an approximate merit function developed in this paper, which is more rigorous than existing heuristic approaches. Some promising results are shown through two case studies, one of which is on a relatively large transportation network. Also, the results obtained can be further applied for real time traffic operations purposes.**

I. Introduction

For decades dynamic traffic assignment (DTA) which can predict, in a short-term fashion, the future dynamic traffic states, has been extensively studied to fulfill various needs of the intelligent transportation systems (ITS). During the past 30 years, two distinct approaches have dominated the methodologies applied to the DTA research: the simulation-based (or microscopic) and the analytical (or macroscopic) approach. In this paper, we focus on the analytical DTA approach, especially the variational inequality (VI) method since it has been proved that VI provides a powerful analytical tool in terms of modeling DTA (Ran and Boyce, 1996; Chen, 1999; Peeta and Ziliouskopolous, 2001). Particularly, we are interested in the so-called dynamic user equilibrium (DUE) which is a special and probably the most challenging problem of DTA.

VI has been applied for long for modeling various traffic interactions for static traffic assignment problems (Smith, 1979; Dafermos, 1980). It had not been used to model DTA until Friesz (1993). Later on, Ran and Boyce (1994, 1996) extensively studied the issues of applying VI to formulate and solve different DTA problems. Although formulated continuously in the temporal domain, the models by Friesz and Ran and Boyce were actually solved by the discretization. However, these discretized models were only treated as certain solution procedure without rigorous investigations on their mathematical properties such as the model and solution properties. Chen (1999) was among the first who specifically focused on the discrete-time VI models for DTA. Wie (2002) studied the discrete-time DUE problem using the nonlinear complementarity problem (NCP) formulation and pointed out the significant difference between the continuous-time and discrete-time DUE models, i.e., the former are infinite dimensional mathematical programming (MP) problems while the latter are finite dimensional MP problems. By integrating the cell transmission model (CTM), Lo and Szeto (2002) proposed the CTM-based DUE model which is preferable in terms of capturing various dynamic phenomena in DTA such as spillbacks. Although normally a non-smooth optimization problem, the CTM-based DUE model has promising performances in terms of computation (Lo and Szeto, 2002). The quasi-variational inequality (QVI) technique was first adopted by Bliemer and Bovy (2003) to model various interactions for the multi-user-class DUE problem.

In this paper, we mainly consider the discrete-time DUE since to date solving the continuous-time DUE model for practical transportation networks is still not feasible (Bliemer and Bovy, 2003). In particular, we found that even for the basic discrete-time DUE which is deterministic, single-user-class and fixed demands, it is still more appropriate to formulate the problem as a QVI rather than a regular VI, provided the dynamics of DUE especially its flow propagation constraints are to be modeled explicitly. For a group of DUE models, the flow propagation constraints can be expressed as follows (Chen, 1999; Bliemer and Bovy, 2003):

$$u_a^A(k) = v_a^A[k + \hat{\tau}_a(k)], \quad (1)$$

where $u_a^A(k)$ and $v_a^A(k)$ denote, respectively, the total (aggregated) link inflow and exit flow rate of link *a* at time interval *k*, $\tau_a(k)$ the link travel time for link *a* at time *k*, and $\hat{\tau}_a(k)$ the integer valued link travel time for link *a* at time *k*. Since $u_a^A(k), v_a^A(k), \forall a, k$ are defining variables of the DUE model and $\tau_a(k)$ is a function of them, the actual expression of equation (1) will vary as long as the defining variables change. This implies that the defining set of the DUE model changes with its variables. Clearly, regular VI can not capture properly such a unique feature of DUE. This observation motivates us to adopt the QVI technique, which is "an extension of a VI in which the defining set of the problem varies with the variable" (Facchinei and Pang, 2003), to study the basic discrete-time DUE.

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To complement our previous work on path-based QVI model for DUE (Ban et al., 2004a and 2004b), we present in this paper a link based QVI model for the basic discrete-time DUE problem. We can further prove that under mild assumptions the solution set of the proposed QVI model is nonempty and bounded, a sharper result compared with that of Bliemer and Bovy (2003) in which only the solution existence is established. In order to solve the QVI model, the damped Newton approach is adopted with two distinct features. Firstly, the sub-problem is formulated as an NCP which is generally much easier to solve than regular VIs. In particular, the PATH algorithm (Dirkse and Ferris, 1995) has been shown to be very efficient to solve these NCP sub-problems. In the second, we develop an approximate merit function for the QVI model based on the integer valued link travel time. Since we can prove that a solution of the QVI model is obtained as long as a zero of the merit function is found, the step size of the iterative solution algorithm can be calculated in a more rigorous way, guided by the approximate merit function, than the existing heuristic methods. Numerical examples show fairly promising results computed using the proposed model and solution algorithm.

This paper is organized as follows. The discrete-time DUE model, especially the QVI formulation, is presented in section 2. The solution existence and boundedness condition for the proposed model is established, as well as the definition of an approximate merit function. Section 3 mainly addresses the solution algorithm issues of the proposed model, including reformulating the sub-problem as an NCP and adopting the Newton-based method to compute the step size. Numerical examples are given in section 4, followed by the concluding remarks and future research directions in section 5.

II. Discrete time DUE model

Most of the VI-based analytical DTA models start with continuous descriptions for the dynamic route choice and dynamic network constraints. For details, we refer to Ran and Boyce (1996), Bliemer and Bovy (2003), and Ban (2005). In order to obtain the discrete-time model, we can evenly divide the entire study period T' into K' time intervals, through introducing the length of each time interval δt such that $K'\delta t = T'$. Under the assumption that the link travel time is constant within each of the time interval, we can define the so-called indicator variable $\delta_a(k',k)$:

$$\delta_a(k',k) = \begin{cases} 1, \text{if } k'+\hat{\tau}_a(k') = k \\ 0, \ othereise \end{cases} . \tag{2}$$

Consequently, we can obtain the discrete-time DUE model as follows (Ban, 2005).

$$\begin{cases} 0 \le u_{as}(k) \perp \{\tau_a(k) + \pi^{h_a s}[k+\hat{\tau}_a(k)] - \pi^{l_a s}(k)\}) \ge 0, \forall a,s,k & \text{(3a)} \\ \sum_{a\in A(j)} u_{as}(k) - [d^{js}(k) + \sum_{a\in B(j)}\sum_{k'} \delta_a(k',k)u_{as}(k')] = 0, \forall j,s,\ j\ne s,k & \text{(3b)} \\ \pi^{js}(k) \ge 0, \forall j,s,\ j \ne s,k & \text{(3c),} \end{cases}$$

where $u_{as}(k)$ denotes the link inflow to link a with respect to destination s at time k (the so-called disaggregated link inflow), $\pi^{js}(k)$ and $d^{js}(k)$ the minimum travel time and trip demands from node j to destination s at time k respectively, and *A(j)* and *B(j)* the set of links entering to and emanating from node j respectively. Further, h_a and l_a denote the head node and tail node of link a, respectively.

A. QVI Formulation

Solving the discrete model (3) directly is still challenging due to the involvement of the indicator variable, although $\delta_a(k',k)$ is actually implicitly determined by the link inflow variable. However, after further investigation, we can show that (3) is equivalent to the following QVI problem, denoted as $QVIA(K^A,\tau)$ (Ban, 2005):

Find $\bar{u}^A \in K^A(\bar{u}^A)$ such that $\forall u^A \in K^A(\bar{u}^A)$,

$$\tau(\bar{u}^A)^T(u^A - \bar{u}^A) \ge 0, \tag{4}$$

where $K^A(\bar{u}^A)$ is a point to set mapping defined as below:

$$K^A(\bar{u}^A) = \{u^A = [u_a^A(k)] \in R^{|A|\cdot K'} \mid \exists u = [u_{as}(k)] \in R^{|A|\cdot|S|\cdot K'} \ge 0, \tag{5a}$$

$$\sum_{a\in A(j)} u_{as}(k) - [d^{js}(k) + \sum_{a\in B(j)}\sum_{k'} \bar{\delta}_a(k',k)u_{us}(k')] = 0, \forall j,s,\ j\ne s,k \tag{5b}$$

$$u_a^A(k) = \sum_{s\in S} u_{as}(k), \forall a,k \tag{5c}$$

$$\}$$

in which $u^A = [u_a^A(k)]_{a\in A, k=1,2,\cdots K'}$ is the vector of the aggregated link inflows , $u = [u_{as}(k)]_{a\in A, s\in S, k=1,2,\cdots K'}$ is the vector of the disaggregated link inflows, and $\bar{\delta}_a(k',k)$ is the indicator variable realized at $\bar{u}^A$ (denoted as the base inflow). Note that N, A, S denote respectively the set of nodes, links and destination nodes in the network. In case (5c) holds for u^A and u, u (u^A) is called the corresponding disaggregated (aggregated) variable of u^A (u). It is obvious from (5) that for a given base inflow $\bar{u}^A$, $\bar{\delta}_a(k',k), \forall a,k',k$ reduces to a fixed value (0 or 1) which implies that $K^A(\bar{u}^A)$ will become a polyhedral set. However, as $\bar{u}^A$ changes, so does $K^A(\bar{u}^A)$. This essentially illustrate the fact that the defining set of $QVIA(K^A,\tau)$ varies with its variable – the reason why (4) is a QVI model (Facchinei and Pang, 2003).

The equivalence between $QVIA(K^A,\tau)$ and the discrete-time DUE (3) can be established in Theorem 1.

Theorem 1 $\bar{u}^A \in K^A(\bar{u}^A)$ solves $QVIA(K^A,\tau)$ if and only if there exists $\bar{u} \in K(\bar{u})$ and $\bar{\pi} \in R^{(|N|-1)\cdot|S|\cdot K'} \ge 0$ such that $(\bar{u},\bar{\pi})$ solves model (3) and $\bar{u}^A$ an $\bar{u}$ are corresponding to each other (i.e., $\bar{u}_a^A(k) = \sum_{s\in S}\bar{u}_{as}(k), \forall a\in A, k = 1,\cdots,K'$).

Proof. See Ban (2005)

B. Solution Existence Condition for the QVI Model

We next need to make sure $QVIA(K^A,\tau)$ has at least one solution. In order to do this, we first impose assumptions on the link travel time function as follows.

Assumption 1 Assume the following statements hold for the link travel time function τ which is a vector function:

a) τ is a continuous and positive function in terms of the aggregated inflow vector $u^A \geq 0$,

b) τ is finite as long as u^A is finite, and

c) τ is a coercive function in terms of u^A, i.e.,

$$\lim_{u^A \to \infty} \frac{(u^A)^T \tau(u^A)}{\|u^A\|} = \infty .$$

With these assumptions in place, we have Theorem 2.

Theorem 2 $QVIA(K^A, \tau)$ has a nonempty and bounded solution set.

Proof. See Ban (2005).

Note that the definition of indicator variable $\delta_a(k', k)$ in equation (2) only involve the integer valued link travel time, the definition of $K^A(\bar{u}^A)$ can thus be determined solely by the integer valued link travel time. Therefore, we have the following theorem.

Theorem 3 For a given base inflow $\bar{u}^A$, assume $\tilde{u}^A$ is a solution to $QVIA(K^A(\bar{u}^A), \tau)$ which is a normal VI and denoted as $VI(K^A(\bar{u}^A), \tau)$. Further denote $\hat{\bar{\tau}}_a(k)$ and $\hat{\tilde{\tau}}_a(k)$ the integer valued link travel time for link *a* at time *k* evaluated, respectively, at $\bar{u}^A$ and $\tilde{u}^A$. Then, $\tilde{u}^A$ is an optimal solution to the QVI model $QVIA(K^A, \tau)$ if and only if $\hat{\bar{\tau}}_a(k) = \hat{\tilde{\tau}}_a(k), \forall a \in A, k = 1, \cdots, K'$.

Proof. See Ban (2005).

According to Theorem 3, we can define an error function *r* for $QVIA(K^A, \tau)$ as follows:

$$r = \left\| \hat{\bar{\tau}} - \hat{\tilde{\tau}} \right\|_1 , \qquad (6)$$

where $\hat{\bar{\tau}}$ and $\hat{\tilde{\tau}}$ denotes, respectively, the vector of the integer-valued link travel times evaluated at $\bar{u}^A$ and $\tilde{u}^A$. Comparing this definition with the normal definition of the merit function for a regular VI problem (Facchinei and Pang, 2003), we can see that *r* defined in (6) can be regarded as an approximate merit function for $QVIA(K^A, \tau)$ because a) *r* is a nonnegative function, and thus if we find a solution to minimize *r* with $r = 0$, it must be a global solution, and b) an optimal solution is obtained for the QVI model if and only if a zero of *r* is achieved as shown in Theorem 3. However, *r* is still not a "true" merit function for $QVIA(K^A, \tau)$ mainly because in (6), certain local information ($\hat{\bar{\tau}}$) is needed.

The reason we define the approximate merit function *r* is to facilitate the design of the solution algorithm for $QVIA(K^A, \tau)$. In particular, *r* can be applied to monitor the convergence of the algorithm, especially to help calculating the step size in each iteration of the solution procedure, as will be demonstrated in the next section.

III. SOLUTION ALGORITHM

QVIs are well-defined mathematical programming problems which were first introduced by Bensoussan and Lions (1973). Since then, progresses have been made from both the theoretical and algorithmic aspects (Outrata and Zowe, 1995; Pang and Fukushima, 2003). However, due to the complexity of QVI, especially its non-convexity nature, the study of QVI to date is still in its infancy (Pang and Fukushima, 2003). In particular, there is still no solution approach which can effectively tackle a general QVI problem. In this paper, we propose a variant of the Newton-based method (Outrata and Zowe, 1995) to iteratively solve $QVIA(K^A, \tau)$, guided by the approximated merit function defined in (23). The algorithm is described as below.

Step 1. Initialization. Assign a feasible base inflow $(\bar{u}^A)^0$.
Step 2. Main Loop. Set *n=0*.
  Step 2.1 Construct current VI sub-problem $VI(K^A(\bar{u}^A)^n, \tau)$ at $(\bar{u}^A)^n$.
  Step 2.2 (**ALG1**) Solve current VI and obtain its solution $(\tilde{u}^A)^n$, denoted as a "candidate" solution.
  Step 2.3 Convergence Test. Check if $r = 0$ at $(\tilde{u}^A)^n$. If yes, go to Step 3; else, go to Step 2.4.
  Step 2.4 (**ALG2**) Determine the step size θ using the damped Newton method.
  Step 2.5 Update and Move. Set $(\bar{u}^A)^{n+1} = (\bar{u}^A)^n + \theta \cdot ((\tilde{u}^A)^n - (\bar{u}^A)^n)$, *n=n+1* and go to Step 2.1.
Step 3. Find an optimal solution $(\tilde{u}^A)^n$.

Fig. 1 Newton-based Solution Algorithm

Obviously, the algorithm listed in Fig.1 is just a simple Newton-based iterative solution process. In order to make it work effectively, two major issues need to be resolved: a) how to solve efficiently the VI sub-problem in Step 2.2, denoted as algorithm **ALG1**, and b) how to construct the step size θ in Step 2.4, denoted as algorithm **ALG2**. These two algorithms will be given later in this section.

A. *NCP Formulation for the VI Sub-Problem*

For a given base inflow $\bar{u}^A$, $QVIA(K^A, \tau)$ reduces to $VI(K^A(\bar{u}^A)^n, \tau)$, a regular VI with a continuous defining function and a polyhedron defining set. As shown in Ban (2005), $VI(K^A(\bar{u}^A)^n, \tau)$ is equivalent to the following NCP:

$$0 \leq u_{as}(k) \perp \{\tau_a(k) + \pi^{h_a s}[k + \hat{\tau}_a(k)] - \pi^{l_a s}(k)\}) \geq 0, \forall a, s, k \qquad (7a)$$

$$0 \leq \pi^{js}(k) \perp \{ \sum_{a \in A(j)} u_{as}(k) - [d^{js}(k) + \sum_{a \in B(j)} \sum_{k'} \bar{\delta}_a(k', k) u_{as}(k')\} \geq 0, \forall j, sS, j \neq s, k \qquad (7b)$$

Then in order to solve the sub-problem $VI(K^A(\bar{u}^A)^n, \tau)$ in Step 2.2 of Fig. 1 (algorithm **ALG1**), we can try to tackle the NCP model (7) instead. The motivation for reformulating a VI as an NCP is that as a special case of the VI, NCP normally has more efficient solution algorithms. In particular, the projection-based methods have been found to be very effective for solving NCPs (Ralph, 1994) because calculating the projection on the nonnegative orthant, the defining set of an NCP, is much easier compared with that on a general convex set. In this paper, we adopt the path search algorithm by Ralph (1994), especially the PATH algorithm by Dirke and Ferris (1995), to solve the NCP model defined in (7). For detailed descriptions of **ALG1**, we refer to Dirkse and Ferris (1995) and Ban (2005).

B. *Calculate the Step Size*

Outrata and Zowe (1995) proposed to compute the step size for a QVI based on its generalized Jacobian matrix. However, due to the involvement of the indicator variable, such a computation is hard to be applied for $QVIA(K^A, \tau)$. For this reason, Bliemer and Bovy (2003) suggested using the MSA (method of successive average) step while calculating the next iterate. Fortunately, because of the approximate merit function *r* defined in equation (6), we can compute the step size in a more rigorous way in this paper. As depicted in Fig. 2, there are three cases if we compare the values of the merit function at the base inflow $\bar{u}^A$ and the candidate solution $\tilde{u}^A$, as well as considering the path from $\tilde{u}^A$ to $\bar{u}^A$.

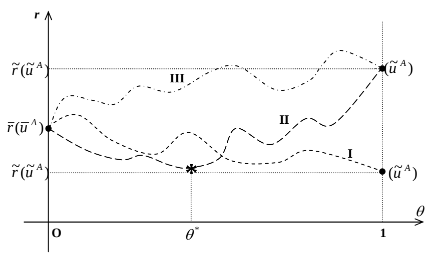

Fig. 2 Calculating the step size

Note that in Fig.2, the horizontal axis is the step size θ and the vertical axis is the value of *r*. Then, θ can be determined using the following damped Newton approach:

If $\tilde{r} < \bar{r}$
  Set $\theta = 1$. That is, the Newton step is taken. (Case I)
else
  Perform a backtracking procedure from $\tilde{u}^A$ to $\bar{u}^A$, using the Armijo sequence, i.e. $\theta = 1, \alpha, \alpha^2, \alpha^3, \cdots, \alpha^m$, where $0 < \alpha < 1$ is the scaling factor and *m* is a positive integer. Evaluate the merit function at each step and denote θ^* the one with the minimum value.
  If $r^* < \bar{r}$, where r^* the value of the merit function evaluated at the step size θ^*
    Set $\theta = \theta^*$. That is, the optimal step is taken. (Case II)
  else
    Set $\theta = 1/n$, where *n* is the current iteration number, i.e. the MSA step is taken. (Case III)

Fig. 3 Damped Newton method for calculating the step size (ALG2)

The approach in Fig.3 basically takes the full step, i.e. set the candidate solution $\tilde{u}^A$ directly as the next iterate, as long as the value of the merit function can be reduced at $\tilde{u}^A$, compared with that at the base inflow $\bar{u}^A$. Otherwise, a backtracking process is performed from $\tilde{u}^A$ to $\bar{u}^A$ using certain Armijo step. If some point between $\tilde{u}^A$ and $\bar{u}^A$ can be found with the merit function value less than that at $\bar{u}^A$, this point will be taken as the next iterate; otherwise, the MSA step will be simply taken. As illustrated in next section, such a method is very effective to reduce the value of *r* for the first several iterations. Further, it is quite efficient to find an optimal or approximate solution to the original problem.

IV. Numerical Examples

In this section, two case studies will be provided to demonstrate the model and solution algorithm proposed in this paper.

A. *Link Travel Time Function*

For illustrative purposes, we deploy a modified version of the dynamic link travel time function by Ran and Boyce (1996) as follows:

$$\tau_a(k) = \frac{60L_a}{U_a}[1 + \alpha(\frac{x_a^A(k)}{L_a k_{ja} l_{na}})^m + \beta(\frac{u_a^A(k)}{\delta t C_{ina} l_{na}})^n], \tag{8}$$

where L_a : length of link *a*, in miles;
U_a : free flow speed of link *a*, MPH;
k_{ja} : jam density of link *a*, vehicles per mile per lane (vpmpl);
C_{ina} : inflow capacity for link *a* , vehicles per minute per lane (vpmpl);
l_{na} : number of lanes for link *a*; and α, m, β, n : positive parameters.

Note that (8) is the so-called whole-link model for constructing the link travel time function for dynamic user equilibrium and has been adopted by Chen (1999) and Wie et al. (2002). Detailed discussions on issues such as the FIFO property for the whole-link model can be found in Carey et al. (2003) and Nie and Zhang (2005).

B. *Case Study I*

This is a small example tested on the hypothetical network depicted in Fig. 4, which is denoted as the D3 network. In the DUE literature, this network was first used by Chen (1999). In this paper, we use slightly different specifications for the D3 network. Particularly, there are two OD pairs, 1→3 and 2→3. Each link is 1 mile in length, except that link 3 is 1.8 miles and link 6 is 2 miles. Other settings of the network can be found in Ban (2005).

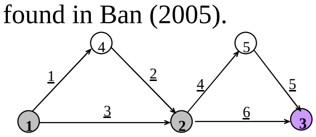

Fig. 4 D3 network

To simulate the fluctuation of traffic volumes during peak hours, we employ a parabolic-shaped curve to represent the OD demands between each OD pair for this case study. In particular, the demands between any OD pair *rs* can be obtained through equation (9):

$$d^{rs}(k) = 10 + 30 * (1 - \left(\frac{k - K/2}{K/2}\right)^2), \forall k, r, s, \tag{9}$$

where *K* denotes the total number of intervals during which OD trips will be generated.

In this case study, we set $K = 8$. The PATH algorithm can solve each NCP sub-problem very efficiently and the Newton-based algorithm converges after 4 iterations. The inflow rate to link 3 (1→2) $u_3(k)$ and its complementary part $\pi^{23}(k+\hat{\tau}_3(k))+\tau_3(k)-\pi^{13}(k)$ are shown in Fig. 5. Note that since we have only one destination (i.e. node 3), the corresponding disaggregated and aggregated variables are exactly the same. From this figure, it is clear that $\min[u_3(k), \pi^{23}(k+\hat{\tau}_3(k))+\tau_3(k)-\pi^{13}(k)]=0, \forall k$ which means that both of the two terms are nonnegative and they are complementary to each other. Therefore, the DUE condition holds for link 3. Similarly, it is easy to check that this relationship holds for all the other five links. Hence, we conclude that we solved the original DUE problem in the sense that the DUE condition holds exactly to the level of 1.e-6 since the convergence criterion is set to 1.0e-6.

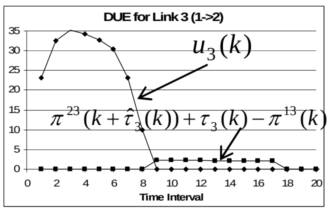

Fig. 5 DUE condition for link 3 for case study I

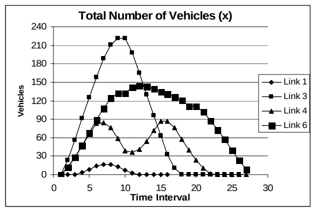

Fig. 6 Total number of vehicles for case study I

Fig. 6, 7 and 8 depict, respectively, the total number of vehicles, link inflow rate and exit flow rate for link 1, 3, 4, and 6 in the D3 network.

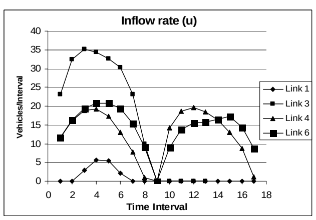

Fig. 7 Link inflow rate for case study I

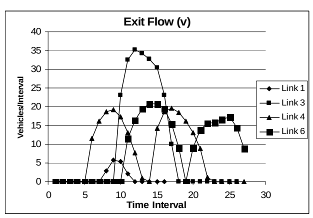

Fig. 8 Link exit flow rate for case study I

C. *Case Study II*

A relatively large example is given in this case study based on the I-394 corridor network in Minneapolis. Fig.9 depicts a map for this network which is near the downtown Minneapolis. The network contains various types of roadways, including freeways, state and county highways, major arterials, and local roads, as marked in dark color in Fig. 9. The coded network based on this map is shown in Fig. 10 which contains 91 nodes and 254 links. We further identified 27 origins (node 1 – 27) and five destinations (node 1, 4, 10, 14, 15). The detailed network data and OD demands data for 48 minutes can be found in Ban (2005).

Fig. 9 A map for the I-394 corridor network

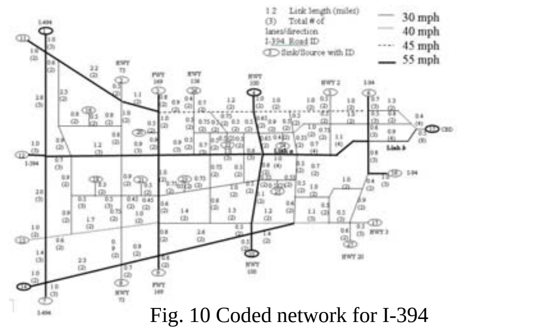

Fig. 10 Coded network for I-394

The Newton-based algorithm converges after 27 iterations and Fig.11 displays the changes of the *r* value with respect to the iteration number. Notice that the value of *r* does not decrease monotonically which is typical for solving DUE due to its non-convexity nature. We claim that an optimal solution is found for this example because i) the algorithm converges to a point with *r* being exactly zero, and ii) for a randomly chosen link, e.g. Link *a* in Fig. 10, the DUE condition holds for this link for all the time intervals, as shown in Fig. 12. Further, this holds for all links.

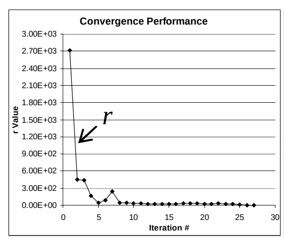

Fig. 11 Convergence performance for I-394

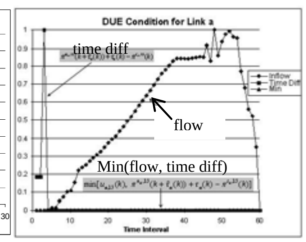

Fig. 12 DUE condition for link *a*

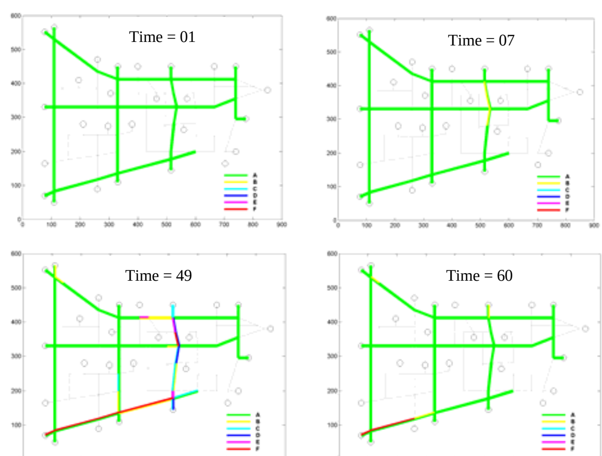

Fig. 13 Time-dependent LOS map for I-394 network

Since we set the stopping criterion for the NCP sub-problem to 1.0e-6, the optimal solution found in both case studies satisfy the DUE condition in the level of 1.0e-6. This essentially implies that the solutions we found are "exact" ones which are quite different from the approximate solutions in the literature (Ran and Boyce, 1996; Chen, 1999; Bliemer and Bovy, 2003). Although one might argue that such accurate solutions are not necessary from a transportation engineer point of view, Boyce et al (2004) has pointed out that exact solutions are necessary for achieving consistent analysis results even for the static traffic assignment. Furthermore, with these accurate solutions in place, we can better facilitate the transportation management and operations, especially in a real-time fashion. For example, Fig. 13 demonstrates the time-dependent level of service (LOS) maps for I-394 network, created using the results from Case Study II. These maps clearly illustrate the changes of the congestion pattern of the network during the entire study period, which can be utilized for predictions of future traffic congestion patterns. Then traffic management agencies can use these predictions for real time traffic management purposes such as developing more effective dynamic route guidance strategies.

V. Conclusion and Future Research

We presented a link based QVI model for the basic discrete-time DUE problem with single-user-class and fixed demands. The solution existence and boundedness condition was established under mild assumptions. By observing that the sub-problem of the QVI model can be formulated and efficiently solved as a well-defined NCP, we applied the Newton-based approach to solve the QVI problem. In order to more rigorously compute the step size at each iteration of the algorithm, an approximate merit function was developed based on the integer-valued link travel time whose zero coincides with the solution of the proposed QVI model. Numerical examples demonstrated that the proposed model and solution approach can produce optimal solutions for even large scale DUE problems.

For future studies, the proposed QVI model especially its solution approaches needs further investigations. Given the non-convexity nature of the QVI formulation, a challenging question would be how to achieve a descent searching direction and then find a step size which can always guarantee the monotone decrease of the approximate merit function. On the other hand, this paper only studied the basic discrete-time DUE problem which is deterministic, single-user-class and fixed demands. Extensions to this basic problem, such as multi-user-class, departure time choice and stochastic route choice, need to be investigated in order to model more realistically the real world scenarios. However, it is the authors' understanding that such extensions can be relatively easily accomplished provided the basic problem can be formulated and solved plausibly.

For years analytical DTA models have been criticized for its relatively simple descriptions about the underlying traffic flow dynamics such as queue accumulating and dissipating. Therefore, a crucial future improvement is to incorporate more reasonable traffic flow models into the proposed QVI formulation. The goal is to develop a DUE model with both the mathematical soundness and plausible descriptions on the underlying traffic flow dynamics. Lastly, we only performed some preliminary investigations on how to utilize the accurate solutions obtained from the model and solution algorithm presented in this paper for real time traffic management and operations. Obviously, more work needs to be done in this aspect to help to improve the effectiveness of the entire ITS.

VI. References

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